THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2050C Mathematical Analysis I

Tutorial 11 (April 29)

Definition. Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$. If there exists a constant $K > 0$ such that

$$|f(x) - f(u)| \leq K|x - u| \qquad \text{for all } x, u \in A, \tag{$*$}$$

then f is said to be a **Lipschitz function** (or to satisfy a **Lipschitz condition**) on A.

Remarks. When A is an interval I, the condition $(*)$ means that the slopes of all line segments joining two points on the graph of $y = f(x)$ over I are bounded by some number K.

Theorem. *If $f : A \to \mathbb{R}$ is a Lipschitz function, then f is uniformly continuous on A.*

Example 1. (a) $f(x) := x^2$ is a Lipschitz function on $[0, b]$, $b > 0$, but does not satisfy a Lipschitz condition on $[0, \infty)$.

(b) $g(x) := \sqrt{x}$ is uniformly continuous on $[0, 2]$ but not a Lipschitz function on $[0, 2]$.

(c) $g(x) := \sqrt{x}$ is uniformly continuous on $[0, \infty)$.

Classwork

1. Let f and g be Lipschitz functions on $[a, b]$. Show that the product fg is also a Lipschitz function on $[a, b]$.

2. Give an example of a Lipschitz function f on $[0, \infty)$ such that its square f^2 is *not* a Lipschitz function.