THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2050C Mathematical Analysis I

Tutorial 1 (January 15)

The following were discussed in the tutorial this week:

1 Negation and Quantifiers

Example 1. Negate the following statements.

(a) n is a prime number between 1 and 10.

(b) If n^2 is divisible by 4, then n is divisible by 2.

(c) For any real number x, $x^2 \geq 0$.

(d) For any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $1/N < \varepsilon$.

2 Order Properties of $\mathbb{R}$

The Order Properties of $\mathbb{R}$. *There is a nonempty subset $\mathbb{P}$ of $\mathbb{R}$, called the set of positive real numbers, that satisfies the following properties:*

(I) $a, b \in \mathbb{P} \implies a + b \in \mathbb{P}$,

(II) $a, b \in \mathbb{P} \implies ab \in \mathbb{P}$,

(III) If $a \in \mathbb{R}$, then exactly one of the following holds:

$$a \in \mathbb{P}, \qquad a = 0, \qquad -a \in \mathbb{P}.$$

Write $a > 0$ if $a \in \mathbb{P}$; and write $a > b$ if $a - b \in \mathbb{P}$.

Example 2. Let $a \in \mathbb{R}$. Show that if $a > 0$, then $1/a > 0$.

Example 3. Let $a, b \in \mathbb{R}$. Show that if $ab > 0$, then either

(i) $a > 0$ and $b > 0$, or

(ii) $a < 0$ and $b < 0$.

3 Solving Simple Inequalities

Example 4. Determine the set $B := \{x \in \mathbb{R} : \dfrac{2x+1}{x+2} < x\}$.

Classwork

1. Negate the following statements.
   (a) For any subset S of $\mathbb{N}$, if $S \neq \emptyset$, then there exists $m \in S$ such that $m \leq n$ for all $n \in S$.
   (b) There exists an $\varepsilon > 0$ such that for any natural number N, $|x_n - x_m| \geq \varepsilon$ for some $n, m \geq N$.
   (c) For any integer $n \geq 3$, there are no three integers a, b, c that satisfy $a^n + b^n = c^n$.

   Solution. (a) There exists a subset S of $\mathbb{N}$ such that $S \neq \emptyset$ and for any $m \in S$ there exists $n \in S$ such that $m > n$.
   (b) For any $\varepsilon > 0$, there exists a natural number N such that $|x_n - x_m| < \varepsilon$ for all $n, m \geq N$.
   (c) There exists an integer $n \geq 3$ such that $a^n + b^n = c^n$ for some integers a, b, c.

   ◀

2. Let $a, b \in \mathbb{R}$. Show that if $0 < a < b$, then $0 < 1/b < 1/a$.

   Solution. Note that $b = (b-a) + a$ and $(b-a), a > 0$, we have $b > 0$. By Example 2, we have $1/b > 0$ and

   $$a > 0, b > 0 \implies ab > 0 \implies 1/(ab) > 0.$$

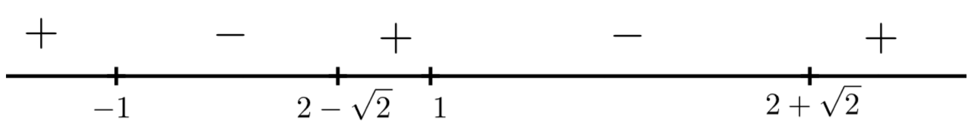
   Hence $1/a - 1/b = (b-a) \cdot 1/(ab) > 0$. Therefore $1/a > 1/b > 0$. ◀

3. Determine the set $C := \left\{ x \in \mathbb{R} : \dfrac{4x-3}{x^2-1} \leq 1 \right\}$.

   Solution. Note that

   $$\begin{aligned} x \in C &\iff 1 - \frac{4x-3}{x^2-1} \geq 0 \iff \frac{x^2-4x+2}{x^2-1} \geq 0 \\ &\iff f(x) := \frac{(x-(2-\sqrt{2}))(x-(2+\sqrt{2}))}{(x+1)(x-1)} \geq 0. \end{aligned}$$

   The sign of $f(x)$ is given by



   Hence $C = \{x \in \mathbb{R} : x < -1\} \cup \{x \in \mathbb{R} : 2-\sqrt{2} \leq x < 1\} \cup \{x \in \mathbb{R} : x \geq 2+\sqrt{2}\}$. ◀