MATH 2050C Mathematical Analysis I
2019-20 Term 2

Hard problems in Chapter 3

3.1-18

Just choose $\epsilon = \frac{x}{2}$, then we can find a natural number $K > 0$ such that $|x_n - x| < \epsilon = \frac{x}{2}$. This will implies

$$x - \frac{x}{2} < x_n < x + \frac{x}{2} \implies \frac{x}{2} < x_n < 2x$$

3.2-12

Note that $0 < \frac{a}{b} < 1 \implies \lim \left(\frac{a}{b}\right)^n = 0$. Then

$$\lim \frac{a^{n+1} + b^{n+1}}{a^n + b^n} = \lim \frac{a\left(\frac{a}{b}\right)^n + b}{\left(\frac{a}{b}\right)^n + 1} = \frac{a \lim \left(\frac{a}{b}\right)^n + b}{\left(\frac{a}{b}\right)^n + 1} = b$$

3.2-17

(a). Example, $x_n = \frac{1}{n}$, then $\lim \frac{n}{n+1} = 1$.
(b). Example, $x_n = \sum_{k=1}^{n} \frac{1}{k}$. We have proofed that (x_n) is divergent. And from following

$$|\frac{x_{n+1}}{x_n} - 1| = \frac{1}{(n+1)x_n} < \frac{1}{n+1}$$

which implies

$$\lim \frac{x_{n+1}}{x_n} = 1$$

3.2-23

We can use the following identify.

$$\max\{a, b\} = \frac{|a-b| + a + b}{2}$$

Since (x_n), (y_n) are all convergent, we know $(x_n - y_n)$ is also convergent, and hence $(|x_n - y_n|)$ is convergent. Hence, we know

$$(\max\{x_n, y_n\}) = \left(\frac{|x_n - y_n| + x_n + y_n}{2}\right)$$

also converges.

Similarly, from the identify

$$\min\{a, b\} = \frac{a + b - |a - b|}{2}$$

we know v_n is convergent.

3.3-5

First, we know $y_2 = \sqrt{p + y_1} > \sqrt{p} = y_1$. We will show that $y_{n+1} > y_n$ for all n by Mathematical Induction. Clearly this is true for $n = 1$. Now let's assume we have the conclusion holds for $n = k - 1$, i.e., $y_k > y_{k-1}$, then we have

$$y_{k+1} = \sqrt{p + y_k} > \sqrt{p + y_{k1}} = y_k$$

So the conclusion is true for $n = k$. So we get (y_n) is an increasing sequence. Hence, we have

$$\sqrt{p + y_n} = y_{n+1} > y_n \implies p + y_n > y_n^2 \implies \frac{1 - \sqrt{1 + 4p}}{2} < y_n < \frac{1 + \sqrt{1 + 4p}}{2}$$

So we can apply Monotone Convergence Theorem to get $y = \lim y_n$ exists. Take limit at both side, we will have

$$y = \sqrt{p + y} \implies y = \frac{1 + \sqrt{1 + 4p}}{2}$$

We can rule out $y = \frac{1-\sqrt{1+4p}}{2}$ since it is negative.

3.3-11

Note that

$$\frac{1}{k^2} < \frac{1}{k(k-1)} = \frac{1}{k-1} - \frac{1}{k}$$

We have

$$x_n = \sum_{k=1}^{n} \frac{1}{k^2} = \frac{1}{1^2} + \sum_{k=2}^{n} \left(\frac{1}{k-1} - \frac{1}{k}\right) = 1 + 1 - \frac{1}{n} < 2$$

Hence, x_n is bounded above by 2. (x_n) is increasing is clear since $x_{n+1} - x_n = \frac{1}{(n+1)^2} > 0$. So x_n is convergent.

3.4-14

We will try to find this subsequence by induction. First, we can find a natural number n_1 with $s - 1 < x_{n_1} \leq s$ by the definition of supremum. Now suppose we have find $n_1 < \cdots < n_k$ such that $s - \frac{1}{l} < x_{n_l} \leq s$ for all $1 \leq l \leq k$, since $s \notin \{x_n : n \in \mathbb{N}\}$, we know $M = \max\{s - \frac{1}{k+1}, x_1, x_2, \ldots, x_{n_k}\} < s$. Hence by the definition of supremum, we can find n_{k+1} with $x_{n_{k+1}} > M$. Clearly we have $n_{k+1} > n_k$ by the choice of M. Then by Mathematical induction, we can find $n_1 < n_2 < \ldots$ such that $s - \frac{1}{k} < x_{n_k} \leq s$. This will clearly show that this subsequence satisfies

$$\lim_{k\to\infty} x_{n_k} = s$$

3.4-15

Clearly, $x_n \in I_n \subset I_1$ will imply (x_n) is a bounded sequence. So we can apply Bolzano-Weierstrass Theorem to get a subsequence (x_{n_k}) which is convergent. Suppose $x = \lim x_{n_k}$, then we will show that $x \in I_n$ for all n. Indeed, for a fixed n, we know that x_{n_k} will satisfy $a_n \leq x_{n_k} \leq b_n$ for all $n_k > n$. Hence, the limit also satisfies $a_n \leq x_{n_k} \leq b_n$, which implies $x \in I_n$ for all n.

3.5-12

First it is easy to find $x_n > 0$ by Mathematical induction. We just note that

$$|x_{n+1} - x_n| = \left|\frac{1}{2+x_n} - \frac{1}{2+x_{n-1}}\right| = \frac{|x_{n-1} - x_n|}{(2+x_n)(2+x_{n-1})} < \frac{1}{4}|x_n - x_{n-1}|$$

So (x_n) is a contractive sequence. Take limit at both side and suppose $x = \lim x_n$, then we have

$$x = \frac{1}{2+x} \implies x = -1 + \sqrt{2} \text{ rule out the negative root since } x_n > 0$$