MATH 2050C Mathematical Analysis I
2019-20 Term 2

3.1-5

(a) Note that $\frac{n}{n^2+1} < \frac{n}{n^2} = \frac{1}{n}$ So for any $\epsilon > 0$, we can choose a natural number K large such that $K > \frac{1}{\epsilon}$. Hence for any $n \geq K$, we have

$$|\frac{n}{n^2+1} - 0| = \frac{n}{n^2+1} < \frac{1}{n} \leq \frac{1}{K} < \epsilon$$

Hence

$$\lim\left(\frac{n}{n^2+1}\right) = 0$$

(d) Note that

$$\left|\frac{n^2-1}{2n^2+3} - \frac{1}{2}\right| = \frac{5}{4n^2+6} < \frac{2}{n^2}$$

So for any $\epsilon > 0$, we choose a natural number K large such that $K > \sqrt{\frac{2}{\epsilon}}$, hence for any $n \geq K$,

$$\left|\frac{n^2-1}{2n^2+3} - \frac{1}{2}\right| < \frac{2}{n^2} \leq \frac{2}{K^2} < \epsilon$$

Hence

$$\lim\left(\frac{n^2-1}{2n^2+3}\right) = \frac{1}{2}$$

3.1-6

(a) For any $\epsilon > 0$, we choose a natural number $K > \frac{1}{\epsilon^2}$, then for any $n \geq K$, we have

$$\left|\frac{1}{\sqrt{n+7}} - 0\right| = \frac{1}{\sqrt{n+7}} < \frac{1}{\sqrt{n}} \leq \frac{1}{\sqrt{K}} < \epsilon$$

Hence,

$$\lim\left(\frac{1}{\sqrt{n+7}}\right) = 0$$

(d) Still for any $\epsilon > 0$, we choose a natural number K with $K > \frac{1}{\epsilon}$. Then for any $n \geq K$, we have

$$\left| \frac{(-1)^n n}{n^2+1} - 0 \right| = \frac{n}{n^2+1} \leq \frac{1}{n} \leq \frac{1}{K} < \epsilon$$

So we get

$$\lim \left(\frac{(-1)^n n}{n^2+1} \right) = 0$$

3.1-8

First, let's show the following,

$$\lim(x_n) = 0 \implies \lim(|x_n|) = 0$$

For any $\epsilon > 0$, we can find a natural number K such that for any $n \geq K$, we have

$$|x_n - 0| < \epsilon$$

by the definition of $\lim(x_n) = 0$. Hence, we have

$$|\ |x_n| - 0| = |\ |x_n|\ | = |x_n| < \epsilon$$

So we have

$$\lim(|x_n|) = 0$$

by the definition of limit.

Second, let's show the reverse also holds,

$$\lim(|x_n|) = 0 \implies \lim(x_n) = 0$$

Again, for any $\epsilon > 0$, we can find a natural number K, such that for any $n \geq K$, we have $|\ |x_n| - 0| < \epsilon$, this is just $|x_n| < \epsilon$. Hence for $n > K$, we have

$$|x_n - 0| = |x_n| < \epsilon$$

Hence

$$\lim(x_n) = 0$$

Example. Consider $(x_n) = ((-1)^n)$. Clearly (x_n) does not converge but $(|x_n|) = (|(-1)^n|) = (1)$, which is a constant sequence and converges.

3.1-10

By definiton, for any $\epsilon > 0$, we can find a natural number K, such that for $n > K$, we have

$$|x_n - x| < \epsilon$$

Now we choose a special ϵ, named $\epsilon = \frac{x}{2}$. $\epsilon > 0$ holds since $x > 0$. So there exists such K, and for any $n \geq k$, we have $|x_n - x| < \frac{x}{2}$. By the properties of absolute values, we have

$$-(x_n - x) < \frac{x}{2}$$

and it implies

$$x_n > \frac{x}{2} > 0$$

holds for all $n \geq K$.

3.1-14

We choose $a = \frac{1}{b} - 1$(which implies $b = \frac{1}{1+a}$). Since $0 < b < 1$, we will get $a > 0$. So we have

$$|nb^n - 0| = \frac{n}{(1+a)^n}$$

By the Binomial Theorem,

$$(1+a)^n = 1 + na + \frac{1}{2}n(n-1)a + \cdots \geq \frac{n(n-1)}{2}a^2$$

Hence, we can choose K with $K > \frac{2}{\epsilon a^2} + 1$, then for $n \geq K$, we have

$$|nb^n - 0| \leq \frac{2n}{n(n-1)a^2} = \frac{2}{(n-1)a^2} \leq \frac{2}{(K-1)a^2} < \epsilon$$

This means

$$\lim(nb^n) = 0$$