MATH 2050C Mathematical Analysis I
2019-20 Term 2

Solution to Problem Set 3

2.4-4(a)

For the infimum part, we show that $b \sup S$ is a lower bound of bS and $u \leq b \sup S$ for any lower bound u of bS.

First, since $\sup S \geq s$ for all $s \in S$ and $b < 0$, $b \sup S \leq bs$ for all $bs \in bS$. Thus $b \inf S$ is a lower bound of bS.

Suppose u is a lower bound of bS, i.e. $u \leq bs$ for all $bs \in bS$. Thus $u/b \geq s$ for all $s \in S$ since $b < 0$. So u/b is an upper bound of S and $u/b \geq \sup S$. We have $u \leq b \sup S$. As u is arbitrary lower bound, it follows that $b \sup S = \inf(bS)$ by the definition.
For the supremum part, the same idea as above.

2.4-7

To show $\sup(A + B) = \sup A + \sup B$, take any element $a + b \in A + B$. Since $a \leq \sup A$ and $b \leq \sup B$, $a+b \leq \sup A+\sup B$. $\sup A+\sup B$ is an upper bound. From Lemma 2.3.4, for any positive ε, there exist $a_\varepsilon \in A, a_\varepsilon + \varepsilon/2 \geq \sup A$ and $b_\varepsilon \in B, b_\varepsilon + \varepsilon/2 \geq \sup B$. Thus $a_\varepsilon + b_\varepsilon + \varepsilon \geq \sup A + \sup B$. $\sup(A + B) = \sup A + \sup B$ by Lemma 2.3.4 again.
For $\inf(A + B) = \inf A + \inf B$, apply similar discussion.

2.4-11

Suppose $A = \{g(y) : y \in Y\}$, $B = \{f(x) : x \in X\}$. We will prove for any $a \in A, b \in B$, we have $a \leq b$.

Indeed, for any $a = g(y_0) \in A$ for some $y_0 \in Y$, $b = f(x_0) \in B$ for some $x_0 \in X$, we have $a = \inf\{h(x, y_0) : x \in X\}$ by definition. Note that $h(x_0, y_0)$ is an element in the set $\{h(x, y_0) : x \in X\}$, we will have

$$\inf\{h(x, y_0) : x \in X\} \leq h(x_0, y_0)$$

Similarly, we will have

$$f(x_0) = \sup\{h(x_0, y) : y \in Y\} \geq h(x_0, y_0)$$

Based on these two formulas, we have $a = g(y_0) \leq h(x_0, y_0) \leq f(x_0) = b$. Then we can apply the result of **Example 2.4.1 (b)** to get

$$\sup A \leq \inf B$$

which is exactly

$$\sup\{g(y) : y \in Y\} \leq \inf\{f(x) : x \in X\}$$

2.4-17

As in **Theorem 2.4.7**, we choose $S := \{s \in \mathbb{R} : 0 \leq s, s^3 < 2\}$. So we also have $1 \in S$ as $1^3 \leq 2$. We note 2 is an upper bound of S, since for any $s \in S$, if $s > 2$, then $s^3 > 8 > 2$, which contradicts the definition of S. So S has a supremum in $\mathbb{R}$. So we suppose $x = \sup S$. We will proof $x^3 = 2$ by Contradiction.

First, let's assume $x^3 < 2$. We notes that

$$\left(x + \frac{1}{n}\right)^3 = x^3 + \frac{3x^2}{n} + \frac{3x}{n^2} + \frac{1}{n^3} \leq x^3 + \frac{1}{n}\left(3x^2 + 3x + 1\right)$$

So by Archimedean Property, we can choose n large enough such that

$$\frac{1}{n} < \frac{2 - x^3}{3x^2 + 3x + 1}$$

since the right hand side is positive. Then we will have

$$\left(x + \frac{1}{n}\right)^3 < x^3 + (2 - x^3) = 2$$

Hence $x + 1/n \in S$, which contradicts that x is an upper bound of S.

Second, we assume $x^3 > 2$. We note

$$\left(x - \frac{1}{n}\right)^3 = x^3 - \frac{3x^2}{n} + \frac{3x}{n^2} - \frac{1}{n^3} \geq x^3 - \frac{3x^2}{n} - \frac{1}{n} = x^3 - \frac{1}{n}(3x^2 + 1)$$

As before, we choose n large enough, such that

$$\frac{1}{n} < \frac{x^3 - 2}{3x^2 + 1}$$

Then we will have

$$\left(x - \frac{1}{n}\right)^3 \geq x^3 - \frac{1}{n}(3x^2 + 1) > x^3 - (x^3 - 2) = 2$$

Hence for any $s \in S$, we will have

$$\left(x - \frac{1}{n}\right)^3 > 2 > x^3$$

which will imply $x - 1/n > s$. This shows $x - 1/n$ is also an upper bound of S. This contradicts with the fact x is a supremum of S.

In conclusion, we have $x^3 = 2$ and finish the proof.

2.5-8

Suppose $\cap_{n=1}^{\infty} J_n \neq \emptyset$ and $x \in \cap_{n=1}^{\infty} J_n$. Thus $x \in J_n, \forall n$ and $x > 0$. By Archimedean property, there exists some $N \in \mathbb{N}$ satisfying $Nx > 1$. Thus $x > \frac{1}{N}$ and $x \notin J_N$. Contradiction.