MATH 2050C Mathematical Analysis I
2019-20 Term 2

Solution to Problem Set 10

5.1-3

We divided $x_0 \in [a, c]$ in three cases.

First, $x_0 \in [a, b)$, For any $\epsilon > 0$, we can find $\delta > 0$, such that if $0 < |x - x_0| < \delta$ with $x \in [a, b]$, then $|f(x) - f(x_0)| < \epsilon$ by the definition of continuous. Then we can choose $\delta' = \min\{\delta, (b - x_0)\}$, then for any $x \in V_{\delta'}(x_0) \cap [a, c]$, we will have $x \in [a, b]$ and $x \in V_\delta(x_0)$, then we will have $|h(x) - h(x_0)| = |f(x) - f(x_0)| < \epsilon$. Hence $\lim_{x \to x_0} h(x) = h(x_0)$.

Second, $x_0 \in (b, c]$. The steps are essentially same with above and we can get $\lim_{x \to x_0} h(x) = h(x_0)$.

At last, if $x_0 = b$, we need to combine the continuouty of f and g. For any $\epsilon > 0$, we can choose δ_1, δ_2 such that for $x \in V_{\delta_1}(b) \cap [a, b]$, we have $|f(x) - f(b)| < \epsilon$, and for $x \in V_{\delta_2}(b) \cap [b, c]$, we have $|g(x) - g(b)| < \epsilon$. Hence we can choose $\delta = \min\{\delta_1, \delta_2\}$, and for any $x \in V_\delta(b) \cap [a, c]$, we will have either $x \in V_\delta(b) \cap [a, b]$ or $x \in V_\delta(b) \cap [b, c]$, which will imply either $|h(x) - h(b)| = |f(x) - f(b)| < \epsilon$ or $|h(x) - h(b)| = |g(x) - g(b)| < \epsilon$. So we will get $|h(x) - h(b)| < \epsilon$ for all $x \in V_\delta(b)$. In conclusion, we have $\lim_{x \to x_0} h(x) = h(x_0)$ for all $x_0 \in [a, c]$ and hence $h(x)$ is continuous on $[a, c]$.

5.1-5

For $x \neq 2$,

$$f(x) = \frac{x^2 + x + 6}{x - 2} = \frac{(x - 2)(x + 3)}{x - 2} = x + 3.$$

To hold the continuity at $x = 2$, by the Remark 1 of Theorem 5.1.2, the necessary and sufficient condition is that $f(2) = \lim_{x \to 2} f(x) = 5$. We conclude that $f(x)$ is continuous at $x = 2$ if and only if define $f(2) = 5$.

5.1-12

Fix $x \in \mathbb{R}$. By The Density Theorem, there exists (x_n) so that

$$x_n \in \mathbb{Q} \cap (x + \frac{1}{n+1}, x + \frac{1}{n}), \quad \forall n \in \mathbb{N}.$$

Thus $\lim(x_n) = x$. Since f is continuous and $f(x_n) = 0, \forall n \in \mathbb{N}$,

$$f(x) = \lim f(x_n) = 0.$$

Since x is arbitrary, we have $f(x) = 0, \forall x \in \mathbb{R}$.

5.2-6

For any $\epsilon > 0$, since g is continuous at b, we can find $\delta > 0$ such that for all $|x - b| < \delta$, we have $|g(x) - g(b)| < \epsilon$. And since $\lim_{x \to c} f = b$, so for $\epsilon' = \delta$, we can find $\delta' > 0$ such that if $0 < |x - c| < \delta'$, we have $|f(x) - b| < \delta$. Now by $|f(x) - b| < \delta$, we have $|g(f(x)) - g(b)| < \epsilon$. In other word, we will have

$$\lim_{x \to c} g(f(x)) = g(b)$$

5.2-8

Denote $h := f - g$. $h(x)$ is continuous on $\mathbb{R}$ by Theorem 5.2.1(a). $h(r) = f(r) - g(r) = 0, \forall r \in \mathbb{Q}$. Apply the result of Exercise 5.1-12. We have $h(x) = 0, \forall x \in \mathbb{R}$, i.e. $f(x) = g(x), \forall x \in \mathbb{R}$.