THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH 2050B Mathematical Analysis I

Tutorial 8 (November 1)

The following were discussed in the tutorial this week:

Continuous Functions

Definition 1. Let $A \subset \mathbb{R}$, let $f : A \to \mathbb{R}$ and let $c \in A$.

- We say that f is **continuous at** c if, given any $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \in A$ satisfying $|x - c| < \delta$, then $|f(x) - f(c)| < \delta$.
- Let $B \subset A$. We say that f is **continuous on** B if f is continuous at every point of B.

Remark. (1) We do not assume that c is a cluster point of A ($c \in A^c$).

Case 1: If $c \in A^c$, then f is continuous at $c \iff \lim_{x\to c} f = f(c)$.

Case 2: If $c \notin A^c$, then $V_\delta(c) \cap A = \{c\}$ for some $\delta > 0$, so that f is automatically continuous at c.

(2) "f is continuous on B" and "$f\big|_B$ is continuous" are different.

Suppose $A \subset \mathbb{R}$, $f : A \to \mathbb{R}$ and $c \in A$.

Sequential Criterion for Continuity. *f is continuous at c if and only if for every sequence (x_n) in A that converges to c, the sequence $(f(x_n))$ converges to $f(c)$.*

Discontinuity Criterion. *f is discontinuous at c if and only if there is a sequence (x_n) in A that converges to c but the sequence $(f(x_n))$ does not converge to $f(c)$.*

Example 1. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and that $f(r) = 0$ for every rational number r. Show that $f(x) = 0$ for all $x \in \mathbb{R}$.

Example 2. Determine the points of continuity of the function $f(x) := [1/x]$, $x \neq 0$. Here $[\,\cdot\,]$ is the greatest integer function defined by

$$[x] := \sup\{n \in \mathbb{Z} : n \leq x\}.$$

Example 3. Give an example for each of the following:

(a) $f : \mathbb{R} \to \mathbb{R}$ is continuous everywhere except at one point.

(b) $f : \mathbb{R} \to \mathbb{R}$ is continuous only at one point.

(c) $f : \mathbb{R} \to \mathbb{R}$ is discontinuous everywhere but $|f|$ continuous everywhere.

(d) $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}\backslash\mathbb{Q}$ but distcontinuous on $\mathbb{Q}$.

Example 4. Is there a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous on $\mathbb{Q}$ but distcontinuous on $\mathbb{R}\backslash\mathbb{Q}$?