MATH 2020 Advanced Calculus II

Tutorial 2

Sep 17 and Sep 19

1. Let R be the region bounded by $y = \sqrt{x}$ and $x = \sqrt{y}$. Express the double integral

$$\iint_R f(x,y)dA$$

using

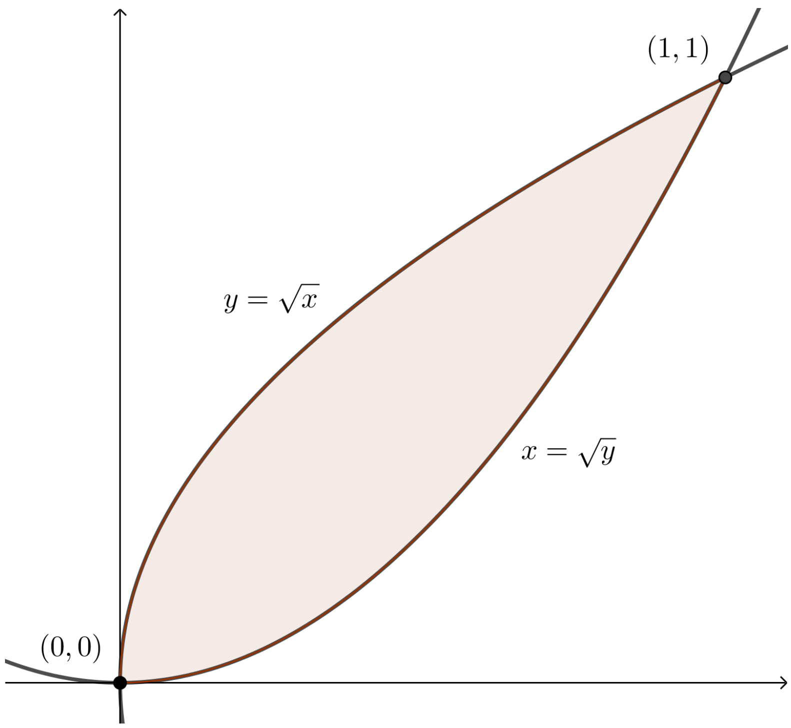
(a) vertical cross-sections;

(b) horizontal cross-sections.

Solution.

(a) $\displaystyle\int_0^1 \int_{x^2}^{\sqrt{x}} f(x,y)dydx.$

(b) $\displaystyle\int_0^1 \int_{y^2}^{\sqrt{y}} f(x,y)dxdy.$



2. Let R be the intersection of the unit disk and the first quadrant, i.e.

$$R = \left\{ (r,\theta) \,\middle|\, r \leqslant 1, 0 \leqslant \theta \leqslant \frac{\pi}{2} \right\}.$$

Evaluate the double integral $\iint_R x dA$ with respect to different orders, i.e. $dydx$ and $dxdy$.

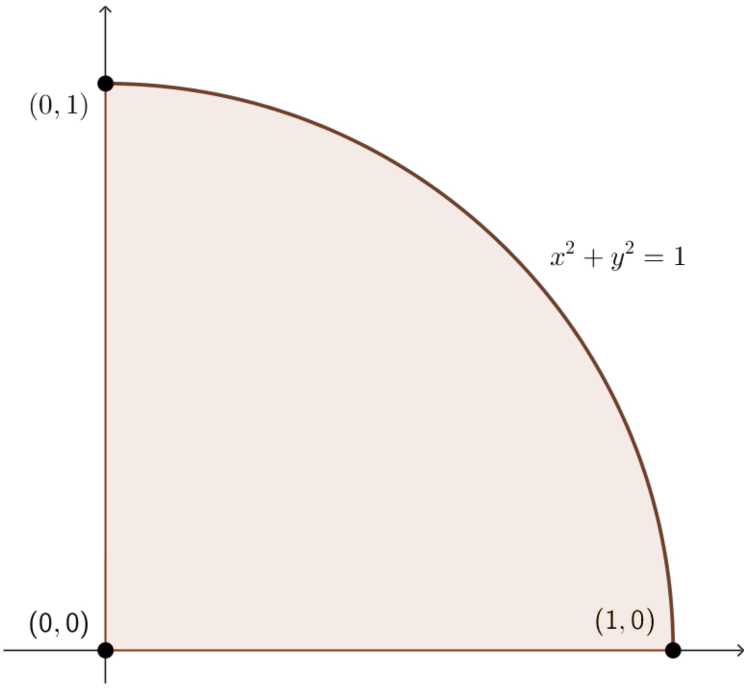
Solution.

$dydx:$

$$\begin{aligned}\iint_R x\,dA &= \int_0^1 \int_0^{\sqrt{1-x^2}} x\,dy\,dx \\ &= \int_0^1 x\sqrt{1-x^2}\,dx \\ &= \left[-\frac{1}{3}\sqrt{1-x^2}^3\right]_0^1 \\ &= \frac{1}{3}.\end{aligned}$$

$dxdy:$

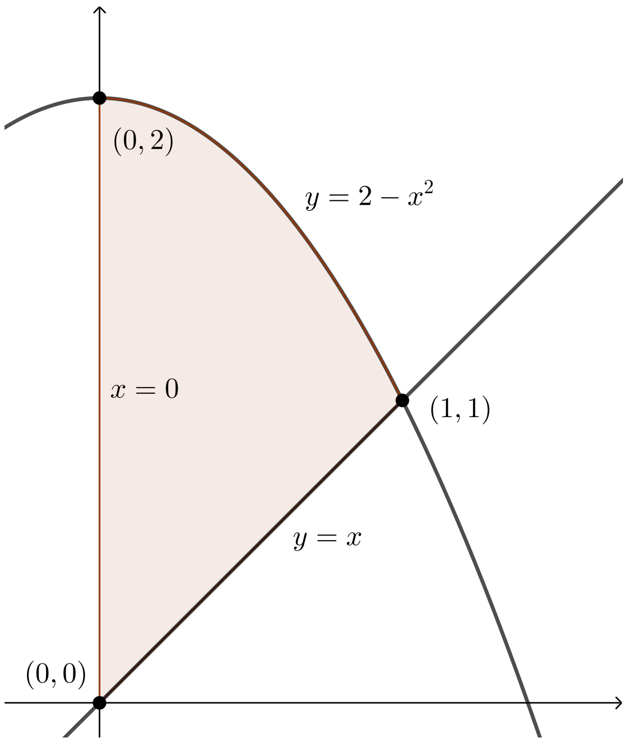
$$\begin{aligned}\iint_R x\,dA &= \int_0^1 \int_0^{\sqrt{1-y^2}} x\,dx\,dy \\ &= \int_0^1 \frac{1}{2}(1-y^2)\,dy \\ &= \frac{1}{2}\left(1-\frac{1}{3}\right) \\ &= \frac{1}{3}.\end{aligned}$$



3. Compute the volume of the solid bounded by $x = 0,\ y = 2 - x^2,\ y = x,\ z = 0$ and $z = x + 10$.

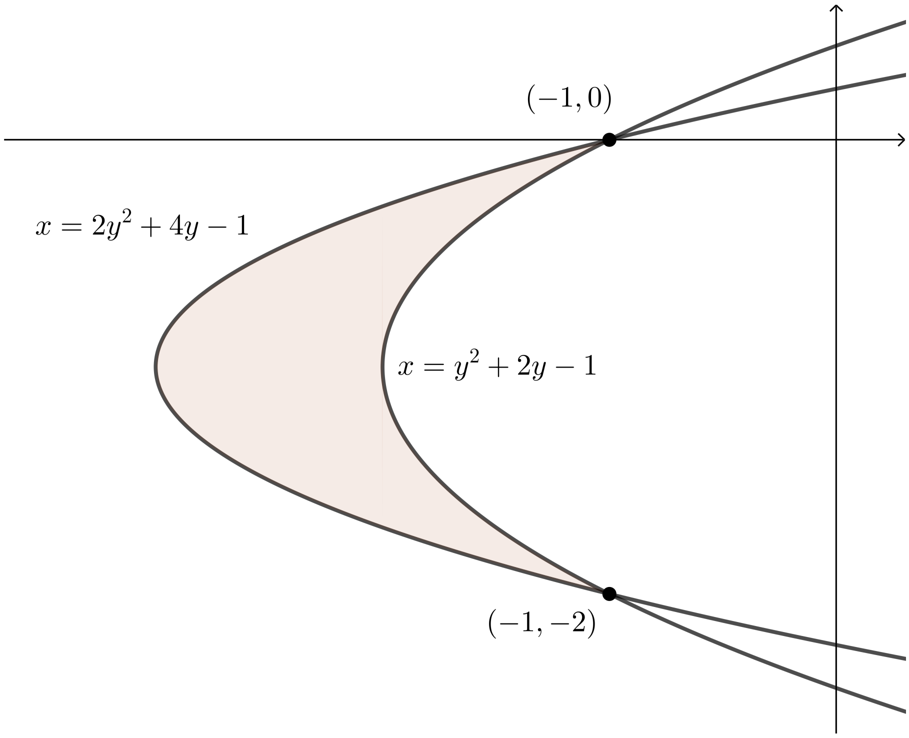
Solution.

$$\begin{aligned}
\text{Volume} &= \int_0^1 \int_x^{2-x^2} (x+10)\,dy\,dx \\
&= \int_0^1 (2-x^2-x)(x+10)\,dx \\
&= \left[(x+10)\left(2x - \frac{x^2}{2} - \frac{x^3}{3}\right)\right]_0^1 - \left[x^2 - \frac{x^3}{6} - \frac{x^4}{12}\right]_0^1 \\
&= \frac{145}{12}.
\end{aligned}$$



4. Compute the area of the region bounded by $x = y^2 + 2y - 1$ and $x = 2y^2 + 4y - 1$.

$$\begin{aligned}
\text{Area} &= \int_{-2}^{0} \int_{2y^2+4y-1}^{y^2+2y-1} dx\,dy \\
&= \int_{-2}^{0} (-y^2 - 2y)\,dy \\
&= \left[-\frac{y^3}{3} - y^2\right]_{-2}^{0} \\
&= 4 - \frac{8}{3} \\
&= \frac{4}{3}.
\end{aligned}$$



5. Describe the region whose area is given by the following expression

$$\int_{-2}^{0}\int_{\frac{x}{2}}^{x+1} dydx + \int_{0}^{1}\int_{2x}^{x+1} dydx.$$

Compute the area as well.

Solution. Let R_1 and R_2 be the regions over which the first and the second integral is taken respectively. Then

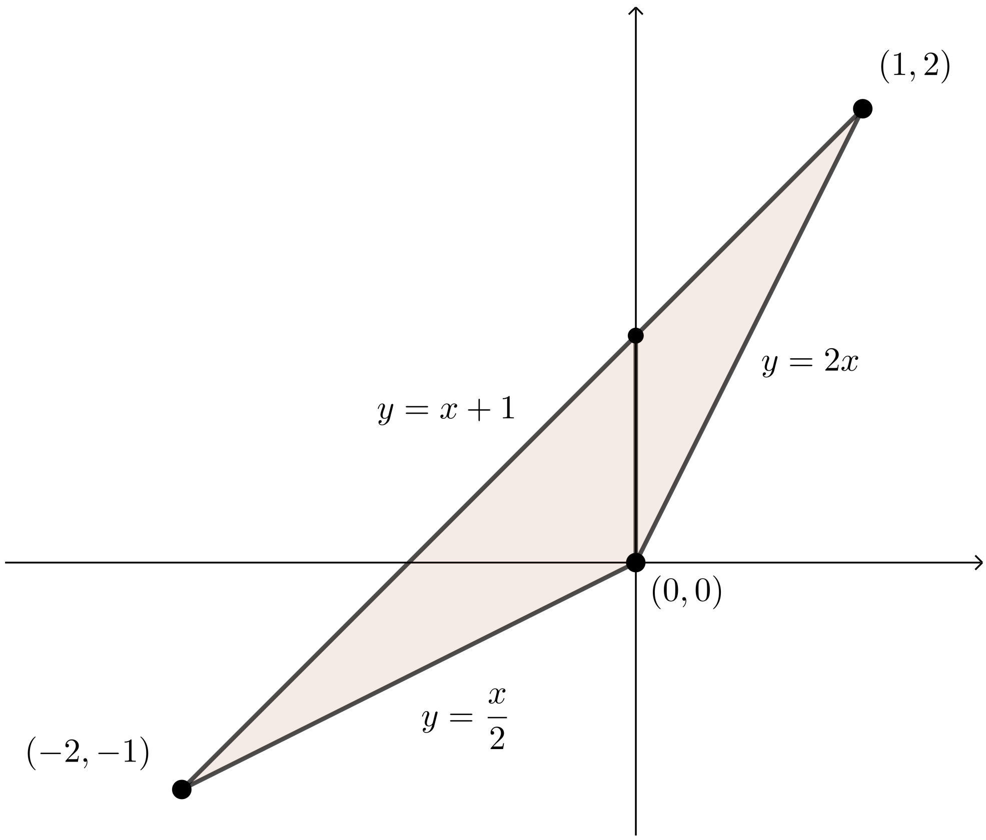
$$R_1 = \left\{(x,y)\,\middle|\, -2 \leqslant x \leqslant 0,\ \frac{x}{2} \leqslant y \leqslant x+1\right\}$$

and

$$R_2 = \{(x,y)|\ 0 \leqslant x \leqslant 1,\ 2x \leqslant y \leqslant x+1\}.$$

We see that their union is the triangle bounded by the lines $y = x+1, y = \dfrac{x}{2}$ and $y = 2x$.

$$\begin{aligned}
\text{Area} &= \int_{-2}^{0}\int_{\frac{x}{2}}^{x+1} dydx + \int_{0}^{1}\int_{2x}^{x+1} dydx \\
&= \int_{-2}^{0}\left(x+1-\frac{x}{2}\right)dx + \int_{0}^{1}(x+1-2x)dx \\
&= \int_{-2}^{0}\left(\frac{x}{2}+1\right)dx + \int_{0}^{1}(1-x)dx \\
&= \left[\frac{x^2}{4}+x\right]_{-2}^{0} + \left[x-\frac{x^2}{2}\right]_{0}^{1} \\
&= -1+2+1-\frac{1}{2} \\
&= \frac{3}{2}.
\end{aligned}$$



(1, 2)
y = 2x
y = x + 1
(0, 0)
y = x/2
(−2, −1)