Vector Fields

Def 10: Let $D \subset \mathbb{R}^2$ or $\mathbb{R}^3$ be a region, then a <u>vector field</u> on D is a mapping $\vec{F} : D \to \mathbb{R}^2$ or $\mathbb{R}^3$ respectively.

In <u>component form</u>:

$\mathbb{R}^2$: $\vec{F}(x,y) = M(x,y)\hat{i} + N(x,y)\hat{j}$

$\mathbb{R}^3$: $\vec{F}(x,y,z) = M(x,y,z)\hat{i} + N(x,y,z)\hat{j} + L(x,y,z)\hat{k}$

where M, N, L are functions on D called the <u>components</u> of $\vec{F}$.

<u>eg 35</u>: $\vec{F}(x,y) = \dfrac{-y\hat{i} + x\hat{j}}{\sqrt{x^2+y^2}}$ on $\mathbb{R}^2 \setminus \{(0,0)\}$

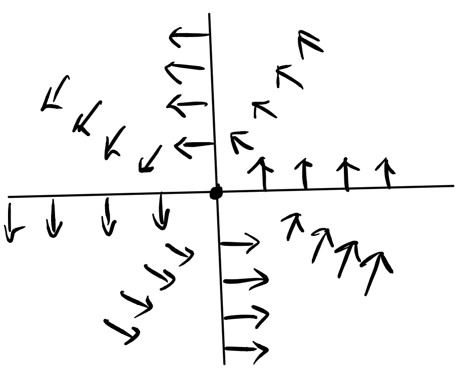
$= -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$ (in polar coordinates)

Properties of $\vec{F}$:

(i) $|\vec{F}(x,y)| = 1$

(ii) $\vec{F} \perp \vec{r}(x,y) = x\hat{i} + y\hat{j}$

$= r(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})$



(Ex: sketch $\vec{r}(x,y) = x\hat{i} + y\hat{j}$)

eg 36 (Gradient vector field of a function)

(i) $f(x,y) = \frac{1}{2}(x^2+y^2)$

$$\vec{\nabla} f(x,y) \overset{\text{def}}{=} \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (x,y) = x\hat{i}+y\hat{j} = \vec{F}(x,y) = \vec{r}$$

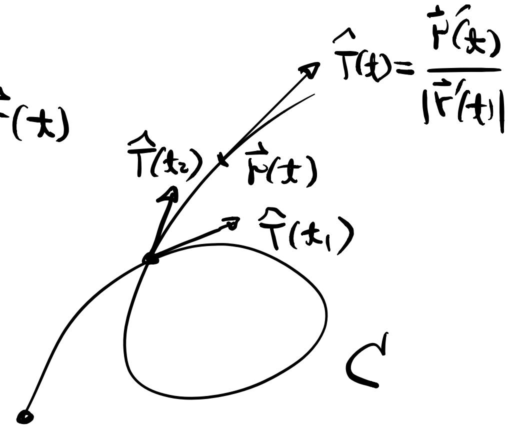
(ii) $f(x,y,z) = x$

$$\vec{\nabla} f(x,y,z) \overset{\text{def}}{=} \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (1,0,0) = \hat{i}$$

eg 37 (Vector field along a curve)

Let C be a curve in $\mathbb{R}^2$ parametrized by

$$\begin{aligned} \vec{r} : [a,b] &\longrightarrow \mathbb{R}^2 \\ t &\longmapsto (x(t), y(t)) = \vec{r}(t) \end{aligned}$$



Recall: $\hat{T}$ = unit tangent vector field along C

$$= \frac{\vec{r}'(t)}{|\vec{r}'(t)|} \quad (\text{provided } \vec{r}'(t) \neq \vec{0})$$

Note that this vector field only defined on C (in general), but not outside C.

Remark: for eg 37.

If we use $ds = |\vec{r}'(t)|dt$, then

$$\hat{T} = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \frac{\frac{d\vec{r}}{dt}}{\frac{ds}{dt}} = \frac{d\vec{r}}{ds} \quad (\text{by Chain rule})$$

where "arc-length s" is defined (up to an additive constant) by

$$s(t) = \int_{t_0}^{t} |\vec{r}\,'(t)|\,dt .$$

A parametrization of a curve C by arc-length s is called <u>arclength parametrization</u>:

$\vec{r}(s)$ = arc-length parametrization

$$\Rightarrow \quad \left| \frac{d\vec{r}}{ds}(s) \right| = 1 .$$

Def 11 A vector field is defined to be <u>continuous</u>/<u>differentiable</u>/<u>C^k</u> if the <u>component functions</u> are.

eg 38: $\begin{cases} \vec{F}(x,y) = \vec{F}(x,y) = x\hat{i} + y\hat{j} \text{ is } C^\infty \\ \vec{F}(x,y) = \dfrac{-y\hat{i} + x\hat{j}}{\sqrt{x^2+y^2}} \text{ is } \underline{\text{not}} \text{ continuous in } \mathbb{R}^2 \end{cases}$

(but <u>continuous</u> in $\mathbb{R}^2 \setminus \{(0,0)\}$)

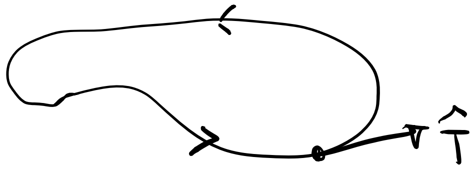
Line integral of vector field

Def 12: Let C be a curve with "<u>orientation</u>" given by a parametrization $\vec{r}(t)$ with $\vec{r}\,'(t) \neq \vec{0}$, $\forall t$. Define the <u>line integral of a vector field $\vec{F}$ along C</u> to be

$$\int_C \vec{F} \cdot \hat{T}\, ds$$

where $\hat{T} = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$ is the unit tangent vector field along C

(i.e. C is oriented in the direction of $\vec{r}'(t)$ or $\hat{T}$ at every point



<u>Note</u>: If $\vec{r}:[a,b]\to\mathbb{R}^n$ ($n=2$ or 3) then

$$\int_C \vec{F}\cdot\hat{T}\,ds = \int_a^b \vec{F}(\vec{r}(t))\cdot\frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}\cdot|\vec{r}\,'(t)|\,dt$$

$$= \int_a^b \vec{F}(\vec{r}(t))\cdot\underbrace{\vec{r}\,'(t)\,dt}_{d\vec{r}}$$

$\therefore$ naturally, we write $\boxed{d\vec{r}=\hat{T}\,ds}$

and $\boxed{\int_C \vec{F}\cdot\hat{T}\,ds = \int_C \vec{F}\cdot d\vec{r}}$

<u>eg 38</u>: $\vec{F}(x,y,z) = z\,\hat{i} + xy\,\hat{j} - y^2\,\hat{k}$

$C: \vec{r}(t) = t^2\,\hat{i} + t\,\hat{j} + \sqrt{t}\,\hat{k}$, $0\le t\le 1$

Then $d\vec{r} = (2t\,\hat{i} + \hat{j} + \frac{1}{2\sqrt{t}}\,\hat{k})\,dt$

and $\int_C \vec{F}\cdot\hat{T}\,ds = \int_C \vec{F}\cdot d\vec{r}$

$$= \int_C (\sqrt{t}\,\hat{i} + t^2 t\,\hat{j} - t^2\,\hat{k})\cdot(2t\,\hat{i} + \hat{j} + \frac{1}{2\sqrt{t}}\,\hat{k})\,dt$$

$$= \int_0^1 (2t\sqrt{t} + t^3 - \frac{t^{3/2}}{2})\,dt = \frac{17}{20}$$ (check!)

Line integral of $\vec{F} = M\hat{i} + N\hat{j}$ along

$C: \vec{r}(t) = g(t)\hat{i} + h(t)\hat{j}$ can be expressed as

$$\int_C \vec{F}\cdot\hat{T}\,ds = \int_C \vec{F}\cdot d\vec{r} = \int_a^b \left(\vec{F}\cdot\frac{d\vec{r}}{dt}\right)dt$$

$$= \int_a^b (Mg' + Nh')\,dt \quad \left(\begin{array}{l}\text{explicitly:} \\ M(g(t),h(t))\,g'(t) \\ \quad + N(g(t),h(t))\,h'(t)\end{array}\right)$$

Note that, usually write $\begin{cases} dx = g'(t)\,dt \\ dy = h'(t)\,dt \end{cases}$

$$\therefore \boxed{\int_C \vec{F}\cdot\hat{T}\,ds = \int_C \vec{F}\cdot d\vec{r} = \int_a^b M\,dx + N\,dy}$$

Similarly for 3-dim

$$\boxed{\int_C \vec{F}\cdot\hat{T}\,ds = \int_C \vec{F}\cdot d\vec{r} = \int_a^b M\,dx + N\,dy + L\,dz}$$