THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS
MATH2010D Advanced Calculus 2019-2020

Solution to Problem Set 9

1. Find the Taylor polynomial of degree 3 generated by $f(x,y)$ at the point $(0,0)$ if $f(x,y) = e^{(x+\sin 2y)}$.

Ans:

$$\begin{aligned}
e^{(x+\sin 2y)} &= e^x \cdot e^{\sin 2y} \\
&= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left[1 + \left((2y) - \frac{(2y)^3}{3!} + \cdots\right) + \frac{((2y) - \cdots)^2}{2!} + \frac{((2y) - \cdots)^3}{3!} + \cdots\right] \\
&= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots\right)(1 + 2y + 2y^2 + \cdots) \\
&= 1 + x + 2y + \frac{x^2}{2} + 2xy + 2y^2 + \frac{x^3}{6} + x^2y + 2xy^2 + \cdots
\end{aligned}$$

Therefore, the required Taylor polynomial is $1 + x + 2y + \frac{x^2}{2} + 2xy + 2y^2 + \frac{x^3}{6} + x^2y + 2xy^2$.

2. Find the Taylor polynomial of degree 6 generated by $f(x,y)$ at the point $(0,0)$ if $f(x,y) = \ln(1 + x\sin y)$.

Ans:

$$\begin{aligned}
&\ln(1 + x\sin y) \\
=\ & (x\sin y) - \frac{(x\sin y)^2}{2} + \frac{(x\sin y)^3}{3} - \cdots \\
=\ & \left(xy - \frac{xy^3}{3!} + \frac{xy^5}{5!} + \cdots\right) - \frac{\left(xy - \frac{xy^3}{3!} + \frac{xy^5}{5!} + \cdots\right)^2}{2} + \frac{\left(xy - \frac{xy^3}{3!} + \frac{xy^5}{5!} + \cdots\right)^3}{3} - \cdots \\
=\ & \left(xy - \frac{xy^3}{3!} + \frac{xy^5}{5!} + \cdots\right) - \left(\frac{x^2y^2}{2} - \frac{x^2y^4}{6} + \cdots\right) + \left(\frac{x^3y^3}{3} + \cdots\right) - \cdots \\
=\ & xy - \frac{xy^3}{6} - \frac{x^2y^2}{2} + \frac{xy^5}{120} + \frac{x^2y^4}{6} + \frac{x^3y^3}{3} + \cdots
\end{aligned}$$

Therefore, the required Taylor polynomial is $xy - \frac{xy^3}{6} - \frac{x^2y^2}{2} + \frac{xy^5}{120} + \frac{x^2y^4}{6} + \frac{x^3y^3}{3}$.

3. (Optional) Let $f(x,y) = e^{x+2y}$.

   (a) Evaluate $\int_0^{1/2}\int_0^{1/2} f(x,y)\,dx\,dy$.

   (b) i. Find the Taylor polynomial $P_2(x,y)$ of degree 2 generated by $f(x,y)$ at the point $(0,0)$.

   ii. Compute $\int_0^{1/2}\int_0^{1/2} P_2(x,y)\,dx\,dy$.
   Is it a good approximation of the integral in (a)? Why?

Ans:

(a)

$$\begin{aligned}\int_0^{1/2}\int_0^{1/2} f(x,y)\,dx\,dy &= \int_0^{1/2}\int_0^{1/2} e^{x+2y}\,dx\,dy\\ &= \int_0^{1/2} [e^{x+2y}]_0^{1/2}\,dy\\ &= \int_0^{1/2} (\sqrt{e}-1)e^{2y}\,dy\\ &= \left[\frac{\sqrt{e}-1}{2}e^{2y}\right]_0^{1/2}\\ &= \frac{(\sqrt{e}-1)(e-1)}{2}\\ &\approx 0.557343\end{aligned}$$

(b) i. We have

$$\begin{aligned}e^{x+2y} &= e^x\cdot e^{2y}\\ &= \left(1+x+\frac{x^2}{2!}+\cdots\right)\left(1+2y+\frac{(2y)^2}{2!}+\cdots\right)\\ &= 1+x+2y+\frac{x^2}{2}+2xy+2y^2+\cdots\end{aligned}$$

Therefore, $P_2(x,y)=1+x+2y+\frac{x^2}{2}+2xy+2y^2$.

ii.

$$\begin{aligned}\int_0^{1/2}\int_0^{1/2} P_2(x,y)\,dx\,dy &= \int_0^{1/2}\int_0^{1/2}\left(1+x+2y+\frac{x^2}{2}+2xy+2y^2\right)dx\,dy\\ &= \int_0^{1/2}\left[x+\frac{x^2}{2}+2xy+\frac{x^3}{6}+x^2y+2xy^2\right]_0^{1/2}dy\\ &= \int_0^{1/2}\frac{31}{48}+\frac{5}{4}y+y^2\,dy\\ &= \left[\frac{31}{48}y+\frac{5}{8}y^2+\frac{1}{3}y^3\right]_0^{1/2}\\ &= \frac{25}{48}\\ &\approx 0.520833\end{aligned}$$

We can see that the above is a good approximation since $P_2(x,y)$ is a good approximation of $f(x,y)$ around the point $(0,0)$.

4. Find the absolute maximum and minimum points of the functions on the given domains.

(a) $f(x,y)=2x^2-4x+y^2-4y+1$ on the triangle bounded by the lines $x=0$, $y=2$ and $y=2x$ in the first quadrant.

(b) $f(x,y)=x^2+xy+y^2-6x+2$ on the rectangle bounded by the lines $x=0$, $x=5$, $y=-3$ and $y=0$.

(c) $f(x,y)=xy$ on the region $D=\{(x,y)\,:\,x\geq 0, y\geq 0 \text{ and } x^2+y^2\leq 4\}$.

Ans:

(a) Firstly, we have $\nabla f(x,y)=(4x-4,2y-4)$, so $\nabla f(x,y)=(0,0)$ when $(x,y)=(1,2)$. Therefore, there is no stationary point in the interior of the triangle and there is a stationary point $(1,2)$ lying on the boundary. Furthermore, the hessian matrix of f is

$$H(x,y)=\begin{bmatrix}4 & 0\\ 0 & 2\end{bmatrix}$$

and so

$$H(1,2) = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}.$$

Note that $\det H(1,2) = 8 > 0$ and $f_{xx}(1,2) = 4 > 0$, so f attains minimum at $(1,2)$ and we have $f(1,2) = -5$.

For the boundary:

- Let $\gamma_1(t) = (0,t)$ for $t \in [0,2]$. We have $f(\gamma_1(t)) = t^2 - 4t + 1$ and so $\frac{d}{dt}f(\gamma_1(t)) = 2t - 4$. Note that $\frac{d}{dt}f(\gamma_1(t)) < 0$ when $0 < t < 2$. Therefore, f attains minimum along γ_1 when $t = 2$ and $f(\gamma_1(2)) = f(0,2) = -3$; f attains maximum along γ_1 when $t = 0$ and $f(\gamma_1(0)) = f(0,0) = 1$.
- Let $\gamma_2(t) = (t,2)$ for $t \in [0,1]$. We have $f(\gamma_2(t)) = 2t^2 - 4t - 3$ and so $\frac{d}{dt}f(\gamma_2(t)) = 4t - 4$. Note that $\frac{d}{dt}f(\gamma_2(t)) < 0$ when $0 < t < 1$. Therefore, f attains minimum along γ_2 when $t = 1$ and $f(\gamma_2(1)) = f(1,2) = -5$; f attains maximum along γ_2 when $t = 0$ and $f(\gamma_2(0)) = f(0,2) = -3$.
- Let $\gamma_3(t) = (t,2t)$ for $t \in [0,1]$. We have $f(\gamma_3(t)) = 6t^2 - 12t + 1$ and so $\frac{d}{dt}f(\gamma_3(t)) = 12t - 12$. Note that $\frac{d}{dt}f(\gamma_3(t)) < 0$ when $0 < t < 1$. Therefore, f attains minimum along γ_3 when $t = 1$ and $f(\gamma_3(1)) = f(1,2) = -5$; f attains maximum along γ_3 when $t = 0$ and $f(\gamma_3(0)) = f(0,0) = 1$.

Therefore, the absolute maximum of f is 1 which is attained at $(0,0)$ and the absolute minimum of f is -5 which is attained at $(1,2)$.

(b) Firstly, we have $\nabla f(x,y) = (2x + y - 6, 2y + x)$, so $\nabla f(x,y) = (0,0)$ when $(x,y) = (4,-2)$. Therefore, $(4,-2)$ is the only stationary point and it lies in the interior of the rectangle.

Furthermore, the hessian matrix of f is

$$H(x,y) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

and so

$$H(4,-2) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

Note that $\det H(1,2) = 3 > 0$ and $f_{xx}(4,-2) = 2 > 0$, so f attains minimum at $(4,-2)$ and we have $f(4,-2) = -10$.

For the boundary:

- Let $\gamma_1(t) = (0,t)$ for $t \in [-3,0]$. We have $f(\gamma_1(t)) = t^2 + 2$ and so $\frac{d}{dt}f(\gamma_1(t)) = 2t$. Note that $\frac{d}{dt}f(\gamma_1(t)) < 0$ when $-3 < t < 0$. Therefore, f attains minimum along γ_1 when $t = 0$ and $f(\gamma_1(0)) = f(0,0) = 2$; f attains maximum along γ_1 when $t = -3$ and $f(\gamma_1(-3)) = f(0,-3) = 11$.
- Let $\gamma_2(t) = (t,0)$ for $t \in [0,5]$. We have $f(\gamma_2(t)) = t^2 - 6t + 2$ and so $\frac{d}{dt}f(\gamma_2(t)) = 2t - 6$. Note that $\frac{d}{dt}f(\gamma_2(t)) < 0$ when $0 < t < 3$ and $\frac{d}{dt}f(\gamma_2(t)) > 0$ when $3 < t < 5$. Therefore, f attains minimum along γ_2 when $t = 3$ and $f(\gamma_2(3)) = f(3,0) = -7$; f attains maximum along γ_2 when $t = 0$, $t = 5$, we have $f(\gamma_2(0)) = f(0,0) = 2$ and $f(\gamma_2(5)) = f(5,0) = -3$.
- Let $\gamma_3(t) = (5,t)$ for $t \in [-3,0]$. We have $f(\gamma_3(t)) = t^2 + 5t - 3$ and so $\frac{d}{dt}f(\gamma_3(t)) = 2t + 5$. Note that $\frac{d}{dt}f(\gamma_3(t)) < 0$ when $-3 < t < -5/2$ and $\frac{d}{dt}f(\gamma_3(t)) > 0$ when $-5/2 < t < 0$. Therefore, f attains minimum along γ_3 when $t = -5/2$ and $f(\gamma_3(-5/2)) = f(5,-5/2) = -37/4$; f attains maximum along γ_3 when $t = 0$, $t = -3$, we have $f(\gamma_3(0)) = f(5,0) = -3$ and $f(\gamma_3(-3)) = f(5,-3) = -9$.
- Let $\gamma_4(t) = (t,-3)$ for $t \in [0,5]$. We have $f(\gamma_4(t)) = t^2 - 9t + 11$ and so $\frac{d}{dt}f(\gamma_4(t)) = 2t - 9$. Note that $\frac{d}{dt}f(\gamma_4(t)) < 0$ when $0 < t < 9/2$. Therefore, f attains minimum along γ_4 when $t = 9/2$ and $f(\gamma_3(9/2)) = f(9/2,-3) = -37/4$; f attains maximum along γ_4 when $t = 0$, $t = 5$, we have $f(\gamma_4(0)) = f(0,-3) = 11$ and $f(\gamma_4(5)) = f(5,-3) = -9$.

Therefore, the absolute maximum of f is 11 which is attained at $(0, -3)$ and the absolute minimum of f is -10 which is attained at $(4, -2)$.

(c) Firstly, we have $\nabla f(x, y) = (y, x)$, so $\nabla f(x, y) = (0, 0)$ when $(x, y) = (0, 0)$. Therefore, there is no stationary point in the interior of the region D and there is a stationary point $(0, 0)$ lying on the boundary.

Furthermore, the hessian matrix of f is

$$H(x, y) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and so

$$H(0, 0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Note that $\det H(0, 0) = -1 < 0$, so $(0, 0)$ is a saddle point of f and we have $f(0, 0) = 0$.

For the boundary:

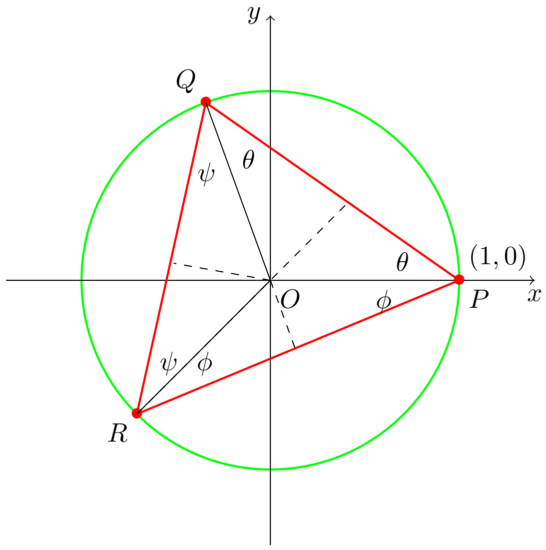
- Let $\gamma_1(t) = (0, t)$ for $t \in [0, 2]$. We have $f(\gamma_1(t)) = 0$ which is a constant function.
- Let $\gamma_2(t) = (t, 0)$ for $t \in [0, 2]$. We have $f(\gamma_2(t)) = 0$ which is a constant function.
- Let $\gamma_3(t) = (2\cos t, 2\sin t)$ for $t \in [0, \pi/2]$. We have $f(\gamma_3(t)) = 4\sin t \cos t = 2\sin 2t$ and so $\frac{d}{dt} f(\gamma_3(t)) = 4\cos 2t$. Note that $\frac{d}{dt} f(\gamma_3(t)) < 0$ when $\pi/4 < t < \pi/2$ and $\frac{d}{dt} f(\gamma_3(t)) > 0$ when $0 < t < \pi/4$. Therefore, f attains minimum along γ_3 when $t = 0$, $t = \pi/2$, we have $f(\gamma_3(0)) = f(2, 0) = 0$ and $f(\gamma_3(\pi/2)) = f(0, 2) = 0$; f attains maximum along γ_3 when $t = \pi/4$ and $f(\gamma_3(\pi/4)) = f(\sqrt{2}, \sqrt{2}) = 2$.

Therefore, the absolute maximum of f is 2 which is attained at $(\sqrt{2}, \sqrt{2})$ and the absolute minimum of f is 0 which is attained at $(t, 0)$ or $(0, t)$ for any $t \in [0, 2]$.

5. Among all triangles with vertices on a given circle, find those that have the largest area.

Ans:

Intuition tells us that the equilateral triangles must have the largest area. However, proving this can be quite difficult unless a good choice of variables in which to set up the problem analytically is made. With a suitable choice of units and axes we can assume the circle is $x^2 + y^2 = 1$ and that one vertex of the triangle is the point P with coordinates $(1, 0)$. Let the other two vertices, Q and R, be as shown in figure below:



Where should Q and R be to ensure that triangle PQR has maximum area?

There is no harm in assuming that Q lies on the upper semicircle and R on the lower, and that the origin O is inside triangle PQR. Let PQ and PR make angles θ and ϕ, respectively, with the negative direction of the x-axis. Clearly $0 \le \theta \le \pi/2$ and $0 \le \phi \le \pi/2$. The lines from O to Q and R make equal angles with the line

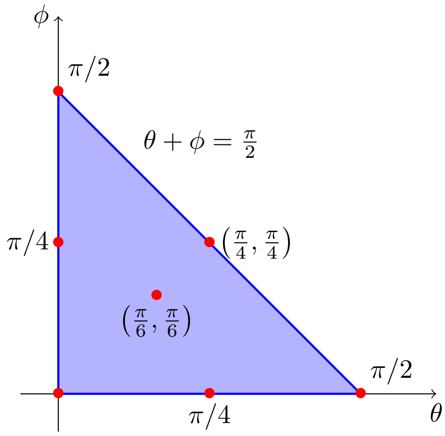
QR, where $2\theta + 2\phi + 2\psi = \pi$. Dropping perpendiculars from O to the three sides of the triangle PQR, we can write the area A of the triangle as the sum of the areas of six small, right-angled triangles:

$$\begin{aligned} A = &\quad 2 \times \tfrac{1}{2} \sin\theta\cos\theta + 2 \times \tfrac{1}{2} \sin\theta\cos\theta + 2 \times \tfrac{1}{2} \sin\psi\cos\psi \\ = &\quad \tfrac{1}{2} \left(\sin 2\theta + \sin 2\phi + \sin 2\psi\right). \end{aligned}$$

Since $2\psi = \pi - 2(\theta + \phi)$, we express A as a function of the two variables θ and ϕ:

$$A = A(\theta, \phi) = \frac{1}{2} \left(\sin 2\theta + \sin 2\phi + \sin 2(\theta + \phi)\right).$$

The domain of A is the triangle $\theta \geq 0$, $\phi \geq 0$, $\theta + \phi \leq \pi/2$. $A = 0$ at the vertices of the triangle and is positive elsewhere. (See the following figure)



The domain of $A(\theta, \phi)$

We show that the maximum value of $A(\theta, \phi)$ on any edge of the triangle is 1 and occurs at the midpoint of that edge. On the edge $\theta = 0$ we have

$$A(0, \phi) = \frac{1}{2} \left(\sin\phi + \sin 2\phi\right) = \sin 2\phi \leq 1 = A(0, \pi/4).$$

Similarly, on $\phi = 0$, $A(\theta, \phi) \leq 1 = A(\pi/4, 0)$. On the edge $\theta + \phi = \pi/2$ we have

$$\begin{aligned} A\left(\theta, \frac{\pi}{2} - \theta\right) &= \tfrac{1}{2} \left(\sin 2\theta + \sin(\pi - 2\theta)\right) \\ &= \sin 2\theta \leq 1 = A\left(\tfrac{\pi}{4}, \tfrac{\pi}{4}\right). \end{aligned}$$

We must now check for any interior critical points of $A(\theta, \phi)$. (There are no singular points.) For critical points we have

$$\begin{aligned} 0 = &\ \tfrac{\partial A}{\partial \theta} = \cos 2\theta + \cos(2\theta + 2\phi), \\ 0 = &\ \tfrac{\partial A}{\partial \phi} = \cos 2\phi + \cos(2\theta + 2\phi), \end{aligned}$$

so the critical points satisfy $\cos 2\theta = \cos\phi$ and, hence $\theta = \phi$. We now substitute this equation into either of the above equations to determine θ:

$$\begin{aligned} \cos 2\theta + \cos 4\theta = &\quad 0 \\ 2\cos^2 2\theta + \cos 2\theta - 1 = &\quad 0 \\ (2\cos 2\theta - 1)(\cos 2\theta + 1) = &\quad 0 \\ \cos 2\theta = \frac{1}{2} \quad \text{or} \quad \cos 2\theta = &\quad -1. \end{aligned}$$

The only solution leading to an interior point of the domain of A is $\theta = \phi = \pi/6$. Note that

$$A\left(\frac{\pi}{6}, \frac{\pi}{6}\right) = \frac{1}{2}\left(\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right) = \frac{3\sqrt{3}}{4} > 1;$$

this interior critical point maximizes the area of the inscribed triangle. Finally, observe that for $\theta = \phi = \pi/6$, we also have $\psi = \pi/6$, so the largest triangle is indeed equilateral.