THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS
MATH2010D Advanced Calculus 2019-2020

Solution to Problem Set 6

1. Let $u(x,y) = \ln(x^3 + y^3 - x^2y - xy^2)$.

   (a) Show that $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} = \dfrac{2}{x+y}$.

   (b) Show that $\dfrac{\partial^2 u}{\partial x^2} + 2\dfrac{\partial^2 u}{\partial x \partial y} + \dfrac{\partial^2 u}{\partial y^2}$ is of the form $-\dfrac{A}{(x+y)^2}$ where A is a constant.

   Ans:

   (a)

$$\begin{aligned}
\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} &= \frac{3x^2 - 2xy - y^2}{x^3 + y^3 - x^2y - xy^2} + \frac{3y^2 - x^2 - 2xy}{x^3 + y^3 - x^2y - xy^2} \\
&= \frac{2x^2 + 2y^2 - 4xy}{x^3 + y^3 - x^2y - xy^2} \\
&= \frac{2(x-y)^2}{(x-y)^2(x+y)} \\
&= \frac{2}{x+y}
\end{aligned}$$

   (b) We have $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} = \dfrac{2}{x+y}$. Then,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} = -\frac{2}{(x+y)^2} \quad \text{and} \quad \frac{\partial^2 u}{\partial y \partial x} + \frac{\partial^2 u}{\partial y^2} = -\frac{2}{(x+y)^2}.$$

   Note that $\dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial^2 u}{\partial y \partial x}$, so

$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = -\frac{4}{(x+y)^2}.$$

2. Let $f(x,y) = x^2 - 3xy + 4y + 1$.

   (a) Find $f(1,1)$, $\dfrac{\partial f}{\partial x}(1,1)$ and $\dfrac{\partial f}{\partial y}(1,1)$.

   (b) Hence, find the equation of tangent plane of $f(x,y)$ at the point $(1,1)$.

   Ans:

   (a) We have $f(1,1) = 3$. Also, $\dfrac{\partial f}{\partial x} = 2x - 3y$ and $\dfrac{\partial f}{\partial y} = -3x + 4$. Therefore, $\dfrac{\partial f}{\partial x}(1,1) = -1$ and $\dfrac{\partial f}{\partial y}(1,1) = 1$.

   (b) The equation of tangent plane of $f(x,y)$ at the point $(1,1)$ is

$$\begin{aligned}
z &= f(1,1) + \frac{\partial f}{\partial x}(1,1)(x-1) + \frac{\partial f}{\partial y}(1,1)(y-1) \\
z &= 3 - (x-1) + (y-1) \\
x - y + z - 3 &= 0
\end{aligned}$$

3. Suppose that all first partial derivatives of the functions $f, g : \mathbb{R}^n \to \mathbb{R}$ exist.

   (a) Show that

$$\nabla[f(\mathbf{x})g(\mathbf{x})] = f(\mathbf{x})\nabla g(\mathbf{x}) + g(\mathbf{x})\nabla f(\mathbf{x}).$$

   (b) If $g(\mathbf{x}) \neq 0$, show that

$$\nabla\left[\frac{f(\mathbf{x})}{g(\mathbf{x})}\right] = \frac{g(\mathbf{x})\nabla f(\mathbf{x}) - f(\mathbf{x})\nabla g(\mathbf{x})}{[g(\mathbf{x})]^2}.$$

Ans:

(a)

$$\begin{aligned}
& \nabla[f(\mathbf{x})g(\mathbf{x})] \\
= & \left(\frac{\partial(f\cdot g)}{\partial x_1}(\mathbf{x}), \frac{\partial(f\cdot g)}{\partial x_2}(\mathbf{x}), \cdots, \frac{\partial(f\cdot g)}{\partial x_n}(\mathbf{x})\right) \\
= & \left(g(\mathbf{x})\frac{\partial f}{\partial x_1}(\mathbf{x}) + f(\mathbf{x})\frac{\partial g}{\partial x_1}(\mathbf{x}), g(\mathbf{x})\frac{\partial f}{\partial x_2}(\mathbf{x}) + f(\mathbf{x})\frac{\partial g}{\partial x_2}(\mathbf{x}), \cdots, g(\mathbf{x})\frac{\partial f}{\partial x_n}(\mathbf{x}) + f(\mathbf{x})\frac{\partial g}{\partial x_n}(\mathbf{x})\right) \\
= & \left(g(\mathbf{x})\frac{\partial f}{\partial x_1}(\mathbf{x}), g(\mathbf{x})\frac{\partial f}{\partial x_2}(\mathbf{x}), \cdots, g(\mathbf{x})\frac{\partial f}{\partial x_n}(\mathbf{x})\right) + \left(f(\mathbf{x})\frac{\partial g}{\partial x_1}(\mathbf{x}), f(\mathbf{x})\frac{\partial g}{\partial x_2}(\mathbf{x}), \cdots, f(\mathbf{x})\frac{\partial g}{\partial x_n}(\mathbf{x})\right) \\
= & f(\mathbf{x})\nabla g(\mathbf{x}) + g(\mathbf{x})\nabla f(\mathbf{x})
\end{aligned}$$

(b)

$$\begin{aligned}
& \nabla[f(\mathbf{x})g(\mathbf{x})] \\
= & \left(\frac{\partial(f/g)}{\partial x_1}(\mathbf{x}), \frac{\partial(f/g)}{\partial x_2}(\mathbf{x}), \cdots, \frac{\partial(f/g)}{\partial x_n}(\mathbf{x})\right) \\
= & \left(\frac{g(\mathbf{x})\frac{\partial f}{\partial x_1}(\mathbf{x}) - f(\mathbf{x})\frac{\partial g}{\partial x_1}(\mathbf{x})}{[g(\mathbf{x})]^2}, \frac{g(\mathbf{x})\frac{\partial f}{\partial x_2}(\mathbf{x}) - f(\mathbf{x})\frac{\partial g}{\partial x_2}(\mathbf{x})}{[g(\mathbf{x})]^2}, \cdots, \frac{g(\mathbf{x})\frac{\partial f}{\partial x_n}(\mathbf{x}) - f(\mathbf{x})\frac{\partial g}{\partial x_n}(\mathbf{x})}{[g(\mathbf{x})]^2}\right) \\
= & \frac{1}{[g(\mathbf{x})]^2}\left[\left(g(\mathbf{x})\frac{\partial f}{\partial x_1}(\mathbf{x}), g(\mathbf{x})\frac{\partial f}{\partial x_2}(\mathbf{x}), \cdots, g(\mathbf{x})\frac{\partial f}{\partial x_n}(\mathbf{x})\right) - \left(f(\mathbf{x})\frac{\partial g}{\partial x_1}(\mathbf{x}), f(\mathbf{x})\frac{\partial g}{\partial x_2}(\mathbf{x}), \cdots, f(\mathbf{x})\frac{\partial g}{\partial x_n}(\mathbf{x})\right)\right] \\
= & \frac{g(\mathbf{x})\nabla f(\mathbf{x}) - f(\mathbf{x})\nabla g(\mathbf{x})}{[g(\mathbf{x}]^2}
\end{aligned}$$

4. Let

$$f(x,y) = \begin{cases} \dfrac{2x^3y}{x^2+2y^2}\cos(xy) & \text{if } \quad (x,y) \neq (0,0); \\ \\ 0 & \text{if } \quad (x,y) = (0,0). \end{cases}$$

(a) Show that f is continuous at $(0,0)$.

(b) Show that $\dfrac{\partial f}{\partial x}(0,0) = 0$ and $\dfrac{\partial f}{\partial y}(0,0) = 0$.

(c) Is f differentiable at (0,0)? Prove your assertion.

Ans:

(a) Note that for $(x,y) \neq (0,0)$, we have

$$\left|\frac{x^3y}{x^2+2y^2}\right| = |x||y|\left|\frac{x^2}{x^2+2y^2}\right| \leq |x||y|\left|\frac{x^2}{x^2+y^2}\right| \leq |x||y|.$$

Therefore, $\left|\dfrac{2x^3y}{x^2+2y^2}\cos(xy)\right| \leq 2|x||y|$, i.e.

$$-2|x||y| \leq \frac{2x^3y}{x^2+2y^2}\cos(xy) \leq 2|x||y|.$$

Also, $\lim\limits_{(x,y)\to(0,0)} -2|x||y| = \lim\limits_{(x,y)\to(0,0)} 2|x||y| = 0$.

By sandwich theorem, we have $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{(x,y)\to(0,0)} \dfrac{2x^3y}{x^2+2y^2}\cos(xy) = 0$.

Therefore, $\lim\limits_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0$ and so $f(x,y)$ is continuous at $(0,0)$.

(b) We have

$$\lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{0}{h^3} = \lim_{h\to 0} 0 = 0.$$

Therefore, $\frac{\partial f}{\partial x}(0,0) = 0$. Also

$$\lim_{h\to 0} \frac{f(0,0+h) - f(0,0)}{h} = \lim_{h\to 0} \frac{0}{2h^3} = \lim_{h\to 0} 0 = 0.$$

Therefore, $\frac{\partial f}{\partial y}(0,0) = 0$.

(c) Note that $\nabla f(0,0) = (0,0)$. Therefore,

$$\lim_{(h_1,h_2)\to(0,0)} \frac{f(0+h_1, 0+h_2) - f(0,0) - \nabla f(0,0)\cdot(h_1,h_2)}{\sqrt{h_1^2+h_2^2}} = \lim_{(h_1,h_2)\to(0,0)} \frac{2h_1^3 h_2}{(h_1^2+2h_2^2)\sqrt{h_1^2+h_2^2}} \cos(h_1 h_2).$$

Note that for $(h_1,h_2) \neq (0,0)$, we have

$$\left| \frac{h_1^3 h_2}{(h_1^2+2h_2^2)\sqrt{h_1^2+h_2^2}} \right| \leq |h_2| \frac{|h_1|}{\sqrt{h_1^2+h_2^2}} \left| \frac{h_1^2}{h_1^2+h_2^2} \right| \leq |h_2|.$$

Therefore, $\left| \frac{h_1^3 h_2}{(h_1^2+2h_2^2)\sqrt{h_1^2+h_2^2}} \cos(h_1h_2) \right| \leq |h_2|$, i.e.

$$-2|h_2| \leq \frac{2h_1^3 h_2}{(h_1^2+2h_2^2)\sqrt{h_1^2+h_2^2}} \cos(h_1 h_2) \leq 2|h_2|.$$

Also, $\lim_{(h_1,h_2)\to(0,0)} -2|h_2| = \lim_{(h_1,h_2)\to(0,0)} 2|h_2| = 0$.

By sandwich theorem, we have $\lim_{(h_1,h_2)\to(0,0)} \frac{h_1^3 h_2}{(h_1^2+2h_2^2)\sqrt{h_1^2+h_2^2}} \cos(h_1 h_2) = 0$.

Therefore, $\lim_{(h_1,h_2)\to(0,0)} \frac{f(0+h_1,0+h_2) - f(0,0) - \nabla f(0,0)\cdot(h_1,h_2)}{\sqrt{h_1^2+h_2^2}} = 0$ and so $f(x,y)$ is differentiable at $(0,0)$.

5. Let

$$f(x,y) = \begin{cases} x\sin\frac{1}{x} + y\sin\frac{1}{y} & \text{if } \ xy \neq 0; \\ 0 & \text{if } \ xy = 0. \end{cases}$$

(a) Show that f is continuous at $(0,0)$.

(b) Show that $\frac{\partial f}{\partial x}(0,0) = 0$ and $\frac{\partial f}{\partial y}(0,0) = 0$.

(c) Is f differentiable at (0,0)? Prove your assertion.

Ans:

(a) Note that for all $(x,y) \in \mathbb{R}^2$ (i.e. no matter $xy \neq 0$ or $xy = 0$), we have $0 \leq |f(x,y)| \leq |x|+|y| \leq 2\sqrt{x^2+y^2}$.
Also, $\lim_{(x,y)\to(0,0)} 0 = \lim_{(x,y)\to(0,0)} 2\sqrt{x^2+y^2} = 0$.
By sandwich theorem, we have $\lim_{(x,y)\to(0,0)} |f(x,y)| = 0$ which implies that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$. Then, $\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0$ and so f is continuous at $(0,0)$.

(b) We have

$$\lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = \lim_{h\to 0} 0 = 0.$$

Therefore, $\frac{\partial f}{\partial x}(0,0) = 0$. Also

$$\lim_{h\to 0} \frac{f(0,0+h) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = \lim_{h\to 0} 0 = 0.$$

Therefore, $\frac{\partial f}{\partial y}(0,0) = 0$.

(c) Note that $\nabla f(0,0) = (0,0)$. Therefore,

$$\lim_{(h_1,h_2)\to(0,0)} \frac{f(0+h_1, 0+h_2) - f(0,0) - \nabla f(0,0)\cdot(h_1,h_2)}{\sqrt{h_1^2+h_2^2}} = \lim_{(h_1,h_2)\to(0,0)} \frac{f(h_1,h_2)}{\sqrt{h_1^2+h_2^2}}.$$

Consider $\gamma(t) = (h_1(t), h_2(t)) = (t,t)$, then

$$\lim_{t\to 0} \frac{f(h_1(t),h_2(t))}{\sqrt{[h_1(t)]^2+[h_2(t)]^2}} = \lim_{t\to 0} \frac{2t\sin\frac{1}{t}}{\sqrt{2t^2}} = \lim_{t\to 0} \sqrt{2}\frac{t}{|t|}\sin\frac{1}{t}$$

which does not exist.

Therefore, $\displaystyle\lim_{(h_1,h_2)\to(0,0)} \frac{f(0+h_1,0+h_2) - f(0,0) - \nabla f(0,0)\cdot(h_1,h_2)}{\sqrt{h_1^2+h_2^2}}$ does not exist and f is not differentiable at $(0,0)$.

6. Let

$$f(x,y) = \begin{cases} x^3\sin\dfrac{1}{x^2} + y^3\sin\dfrac{1}{y^2} & \text{if } \; xy \neq 0; \\ \\ 0 & \text{if } \; xy = 0. \end{cases}$$

(a) Write down $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ explicitly.

(b) Show that $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ are not continuous at $(0,0)$.

(c) Prove that f differentiable at (0,0).

Ans:

(a) For $(x,y) \in \mathbb{R}^2$ such that $xy \neq 0$, it is clear that $\dfrac{\partial f}{\partial x} = 3x^2\sin\dfrac{1}{x^2} - 2\cos\dfrac{1}{x^2}$.

For $(x,0) \in \mathbb{R}^2$, we have

$$\lim_{h\to 0}\frac{f(x+h,0)-f(x,0)}{h} = \lim_{h\to 0}\frac{0-0}{h} = \lim_{h\to 0} 0 = 0$$

and so $\dfrac{\partial f}{\partial x}(x,0) = 0$.

For $(0,y) \in \mathbb{R}^2$ where $y \neq 0$, we have

$$\lim_{h\to 0}\frac{f(h,y)-f(0,y)}{h} = \lim_{h\to 0}\frac{\left(h^3\sin\frac{1}{h^2} + y^3\sin\frac{1}{y^2}\right) - (0)}{h} = \lim_{h\to 0} h^2\sin\frac{1}{h^2} = 0$$

and so $\dfrac{\partial f}{\partial x}(0,y) = 0$.

Therefore, we have

$$\frac{\partial f}{\partial x}(x,y) = \begin{cases} 3x^2\sin\dfrac{1}{x^2} - 2\cos\dfrac{1}{x^2} & \text{if } \; xy \neq 0; \\ \\ 0 & \text{if } \; xy = 0. \end{cases}$$

Similarly,

$$\frac{\partial f}{\partial y}(x,y) = \begin{cases} 3y^2\sin\dfrac{1}{y^2} - 2\cos\dfrac{1}{y^2} & \text{if } \; xy \neq 0; \\ \\ 0 & \text{if } \; xy = 0. \end{cases}$$

(b) Consider $\gamma(t) = (t,t)$. Then, we have

$$\lim_{t\to 0}\frac{\partial f}{\partial x}(\gamma(t)) = \lim_{t\to 0} 3t^2\sin\frac{1}{t^2} - 2\cos\frac{1}{t^2}$$

which does not exists. Then, we have $\displaystyle\lim_{(x,y)\to(0,0)}\frac{\partial f}{\partial x}(x,y)$ does not exist and so $\dfrac{\partial f}{\partial x}$ is not continuous at $(0,0)$. Similarly, we can show that $\dfrac{\partial f}{\partial y}$ is not continuous at $(0,0)$.

(c) Note that $|f(x,y)| \le |x^3| + |y^3| \le 2(\sqrt{x^2+y^2})^3$ for all $(x,y) \in \mathbb{R}^2$ and $\nabla f(0,0) = (\frac{\partial f}{\partial x}(0,0), \frac{\partial f}{\partial y}(0,0)) = (0,0)$. Then,

$$0 \le \left| \frac{f(0+h_1, 0+h_2) - f(0,0) - \nabla f(0,0)\cdot(h_1,h_2)}{\sqrt{h_1^2+h_2^2}} \right| = \left| \frac{f(h_1,h_2)}{\sqrt{h_1^2+h_2^2}} \right| \le \frac{2(\sqrt{h_1^2+h_2^2})^3}{\sqrt{h_1^2+h_2^2}} = 2(h_1^2+h_2^2).$$

Also, $\lim\limits_{(h_1,h_2)\to(0,0)} 0 = \lim\limits_{(h_1,h_2)\to(0,0)} 2(h_1^2+h_2^2) = 0$.
By sandwich theorem, we have

$$\lim_{(h_1,h_2)\to(0,0)} \frac{f(0+h_1, 0+h_2) - f(0,0) - \nabla f(0,0)\cdot(h_1,h_2)}{\sqrt{h_1^2+h_2^2}} = 0.$$

Therefore, f is differentiable at $(0,0)$.