THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS
MATH2010D Advanced Calculus 2019-2020

Problem Set 8

1. Let $f(x, y) = xy$.
   (a) Draw the level set $L_1(f)$.
   (b) Find ∇f and draw ∇f restricted on $L_1(f)$.

2. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a function such that all second partials of f are continuous. Suppose that $\mathbf{v} = (v_1, v_2, v_3)$ is a unit vector, express $\nabla_{\mathbf{v}}(\nabla_{\mathbf{v}} f)$ in terms of the components of $\mathbf{v}$ and the second partials of f.

   What is the interpretation of this quantity for a moving observer?

3. Find the Taylor series generated by the following functions at given points and write down your answers in summation notation.
   (a) $f(x) = \cos x$ at $x = \pi/2$;
   (b) $f(x) = \ln(1 + x)$ at $x = 0$;
   (c) $f(x) = e^x$ at $x = 1$.

4. By considering the Taylor series generated by e^x and $\cos x$ at $x = 0$, find the Taylor polynomials of degree 3 generated by the following functions at $x = 0$.
   (a) $e^x \cos x$;
   (b) $e^{\cos x}$;
   (c) $\dfrac{e^x}{\cos x}$.

5. (a) Find the Taylor polynomial $P_2(x)$ of degree 2 generated by the function $\sqrt[3]{1 + x}$.
   (b) Hence, approximate $\sqrt[3]{1.3}$ and show that the error of your approximation is less that 2×10^{-3}.

6. Let $f(x) = \ln(1 - x)$ for $x < 1$.
   (a) Find the Taylor series generated by $f(x)$ at $x = 0$.
   (b) Write down the Taylor polynomial $T_3(x)$ of degree 3 generated by $f(x)$ at $x = 0$ and the Lagrange remainder $E_3(x)$.
   (c) Hence, approximate $\ln 0.9$ and show that the error of your approximation is less than $\dfrac{1}{4 \times 9^4}$.

7. Let $f(x)$ is a polynomial of degree $n > 0$ and let $a \in \mathbb{R}$.
   (a) If $P_n(x)$ is the Taylor polynomial of degree n generated by $f(x)$ at $x = a$, show that $f(x) = P_n(x)$.
   (b) Suppose that $f(a) = f'(a) = \cdots = f^{(r-1)}(a) = 0$ and $f^{(r)}(a) \neq 0$, where $1 \leq r \leq n$.
   Prove that $(x - a)$ is a factor of $f(x)$ with multiplicity r, i.e. $f(x) = (x - a)^r g(x)$ for some polynomial $g(x)$ such that $g(x)$ is not divisible by $x - a$.
   (c) By using the result in (b), factorize $x^5 - 7x^4 + 19x^3 - 25x^2 + 16x - 4$.