THE CHINESE UNIVERSITY OF HONG KONG

DEPARTMENT OF MATHEMATICS

MATH2010D Advanced Calculus 2019-2020

Problem Set 6

1. Let $u(x,y) = \ln(x^3 + y^3 - x^2y - xy^2)$.
   (a) Show that $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} = \dfrac{2}{x+y}$.
   (b) Show that $\dfrac{\partial^2 u}{\partial x^2} + 2\dfrac{\partial^2 u}{\partial x \partial y} + \dfrac{\partial^2 u}{\partial y^2}$ is of the form $-\dfrac{A}{(x+y)^2}$ where A is a constant.
2. Let $f(x,y) = x^2 - 3xy + 4y + 1$.
   (a) Find $f(1,1)$, $\dfrac{\partial f}{\partial x}(1,1)$ and $\dfrac{\partial f}{\partial y}(1,1)$.
   (b) Hence, find the equation of tangent plane of $f(x,y)$ at the point $(1,1)$.
3. Suppose that all first partial derivatives of the functions $f, g : \mathbb{R}^n \to \mathbb{R}$ exist.
   (a) Show that
   $$\nabla[f(\mathbf{x})g(\mathbf{x})] = f(\mathbf{x})\nabla g(\mathbf{x}) + g(\mathbf{x})\nabla f(\mathbf{x}).$$
   (b) If $g(\mathbf{x}) \neq 0$, show that
   $$\nabla\left[\frac{f(\mathbf{x})}{g(\mathbf{x})}\right] = \frac{g(\mathbf{x})\nabla f(\mathbf{x}) - f(\mathbf{x})\nabla g(\mathbf{x})}{[g(\mathbf{x})]^2}.$$
4. Let
   $$f(x,y) = \begin{cases} \dfrac{2x^3y}{x^2+2y^2}\cos(xy) & \text{if} \quad (x,y) \neq (0,0); \\ 0 & \text{if} \quad (x,y) = (0,0). \end{cases}$$
   (a) Show that f is continuous at $(0,0)$.
   (b) Show that $\dfrac{\partial f}{\partial x}(0,0) = 0$ and $\dfrac{\partial f}{\partial y}(0,0) = 0$.
   (c) Is f differentiable at (0,0)? Prove your assertion.
5. Let
   $$f(x,y) = \begin{cases} x\sin\dfrac{1}{x} + y\sin\dfrac{1}{y} & \text{if} \quad xy \neq 0; \\ 0 & \text{if} \quad xy = 0. \end{cases}$$
   (a) Show that f is continuous at $(0,0)$.
   (b) Show that $\dfrac{\partial f}{\partial x}(0,0) = 0$ and $\dfrac{\partial f}{\partial y}(0,0) = 0$.
   (c) Is f differentiable at (0,0)? Prove your assertion.
6. Let
   $$f(x,y) = \begin{cases} x^3\sin\dfrac{1}{x^2} + y^3\sin\dfrac{1}{y^2} & \text{if} \quad xy \neq 0; \\ 0 & \text{if} \quad xy = 0. \end{cases}$$
   (a) Write down $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ explicitly.
   (b) Show that $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ are not continuous at $(0,0)$.
   (c) Prove that f differentiable at (0,0).