THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS
MATH2010D Advanced Calculus 2019-2020

Problem Set 5

1. Can the function $f(x,y)=\dfrac{\sin x\sin^3 y}{1-\cos(x^2+y^2)}$ be defined at $(0,0)$ in such a way that it becomes continuous there?

2. Let D be a path connected subset of $\mathbb{R}^n$ and let $f:D\to\mathbb{R}$ be a continuous function.

   Suppose that $\mathbf{a},\mathbf{b}\in D$ such that $f(\mathbf{a})<f(\mathbf{b})$.

   Show that for all $L\in\mathbb{R}$ with $f(\mathbf{a})<L<f(\mathbf{b})$, there exists $\mathbf{c}\in D$ such that $f(\mathbf{c})=L$.

3. Let $f:\mathbb{R}^2\to\mathbb{R}$ be a function and let $(a,b)\in\mathbb{R}^2$. We define two single variable functions $g(x)=f(x,b)$ and $h(y)=f(a,y)$.

   (a) If $g(x)$ is continuous at $x=a$ and $h(y)$ is continuous at $y=b$, does it follow that f is continuous at (a,b)? Why?

   (b) If $f(x,y)$ is continuous at (a,b), does it follow that $g(x)$ is continuous at $x=a$ and $h(y)$ is continuous at $y=b$? Why?

4. Let $f(x,y)=\sqrt{2x+3y-1}$. Using the limit definition, find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ at $(-2,3)$.

5. Let $f(x,y,z)=xy+yz+zx$. Using the limit definition, find the directional derivative of f at the point $\mathbf{u}=(1,-1,1)$ along the direction $\mathbf{v}=\mathbf{i}+2\mathbf{j}+\mathbf{k}$.

6. Let $f:\mathbb{R}^2\to\mathbb{R}^2$ be a function defined by

$$f(x,y)=\begin{cases}\dfrac{\sin(x^3+y^4)}{x^2+y^2} & \text{if} \quad (x,y)\neq(0,0);\\ 0 & \text{if} \quad (x,y)=(0,0).\end{cases}$$

   Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ at $(0,0)$.

7. Let $f:\mathbb{R}^2\to\mathbb{R}^2$ be a function defined by

$$f(x,y)=\begin{cases}xy\dfrac{x^2-y^2}{x^2+y^2} & \text{if} \quad (x,y)\neq(0,0);\\ 0 & \text{if} \quad (x,y)=(0,0).\end{cases}$$

   (a) Show that $\dfrac{\partial f}{\partial y}(x,0)=x$ for all $x\in\mathbb{R}$ and $\dfrac{\partial f}{\partial x}(0,y)=-y$ for all $y\in\mathbb{R}$.

   (b) Show that $\dfrac{\partial^2 f}{\partial x\partial y}(0,0)\neq\dfrac{\partial^2 f}{\partial y\partial x}(0,0)$.

8. Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ if

   (a) $f(x,y)=\tan^{-1}\left(\dfrac{y}{x}\right)$

   (b) $f(x,y)=e^{xy}\ln y$

9. Find all first partial derivatives if $f(x,y,z)=\sin^{-1}(x^2+y^2z)$.

10. If $f(x,y)=x\cos y+ye^x$, find all the second-order derivatives, i.e. $\dfrac{\partial^2 f}{\partial x^2}$, $\dfrac{\partial^2 f}{\partial y^2}$, $\dfrac{\partial^2 f}{\partial x\partial y}$ and $\dfrac{\partial^2 f}{\partial y\partial x}$.