THE CHINESE UNIVERSITY OF HONG KONG

DEPARTMENT OF MATHEMATICS

MATH2010C/D Advanced Calculus 2019-2020

Assignment 1, Due Date: 23 Jan, 2020

1. In $\triangle ABC$, $\overrightarrow{AB} = 4\mathbf{i} + 4\mathbf{j}$, $\overrightarrow{AC} = -12\mathbf{i} + 8\mathbf{j}$ and points P, Q lie on BC such that $BP : PQ : QC = 1 : 2 : 1$.

   Find $\angle PAQ$.

2. Let $A = (4, 3, 6)$, $B = (-2, 0, 8)$ and $C = (1, 5, 0)$ be points in $\mathbb{R}^3$.

   Show that $\triangle ABC$ is a right-angled triangle.

3. Suppose that $\mathbf{m}, \mathbf{n} \in \mathbb{R}^n$, where $|\mathbf{m}| = 2$, $|\mathbf{n}| = 1$ and the angle between $\mathbf{m}$ and $\mathbf{n}$ is $\dfrac{2\pi}{3}$.

   If $\mathbf{p} = 3\mathbf{m} + 4\mathbf{n}$ and $\mathbf{q} = 2\mathbf{m} - \mathbf{n}$, find

   (a) $\mathbf{m} \cdot \mathbf{n}$,

   (b) $|\mathbf{p}|$ and $|\mathbf{q}|$,

   (c) the area of the parallelogram spanned by $\mathbf{p}$ and $\mathbf{q}$.

4. Suppose that A, B and C are points on $\mathbb{R}^2$ such that $OABC$ is a kite with $OA = OC$ and $AB = CB$. Let $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ be $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively.

   (a) Express $\overrightarrow{AB}$ and $\overrightarrow{CB}$ in terms of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

   (b) By considering $AB = CB$, show that $\mathbf{b} \cdot \mathbf{a} = \mathbf{b} \cdot \mathbf{c}$.

   (c) Hence, show that $OB \perp AC$.

5. Let $\overrightarrow{OA} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$, $\overrightarrow{OB} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$, $\overrightarrow{OC} = 5\mathbf{i} + \mathbf{j} + 3\mathbf{k}$.

   (a) Find $\overrightarrow{AB} \times \overrightarrow{AC}$.

   (b) Find the volume of tetrahedron $OABC$.

   (Hint: Its volume equals to $\dfrac{1}{6}\times$volume of parallelotope spanned by $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$.)

   (c) By (a) and (b), find the distance from O to $\triangle ABC$.

6. Given $A = (3, -1, 3)$, $B = (0, 7, -2)$ and $C = (-9, 3, -3)$ be three points in $\mathbb{R}^3$.

   (a) Find the coordinates of a point D if AC, BD are perpendicular and AD, BC are parallel.

   (b) i. Find $\angle DCB$.

   ii. Show that A, B, C, D are coplanar (i.e. lying on a same plane) and find the equation of the plane which contains them.

   iii. Show that $ABCD$ is a square and find the area of it.

   (c) $VABCD$ is a pyramid with base $ABCD$. If $V = (12, -14, -12)$,

   i. find the volume of the pyramid;

   ii. find the angle between the plane VAB and the base.

7. Suppose that $L_1 : x+1 = \frac{y-2}{-2} = \frac{z+3}{2}$ and $L_2 : \frac{x-1}{-1} = \frac{y+2}{2} = \frac{z-6}{3}$ are two straight lines.

   (a) Show that L_1 and L_2 intersect each other at one point and find the point of intersection.

   (b) Find the acute angle between L_1 and L_2.

   (c) Find the equation of plane containing L_1 and L_2.

8. Let $\Pi_1 : x - 2y + 2z = 0$ and $\Pi_2 : 3x + y + 2z = 4$ be two planes and let $P(1, 2, -1)$ be a point in $\mathbb{R}^3$.

   (a) Find the angle between Π_1 and Π_2.

   (b) Find the equation of the line passing through the point P which is parallel to the intersection line of the planes Π_1 and Π_2.

9. Let $A = (1, 1, 0)$, $B = (0, 1, 1)$ and $C = (1, -1, 1)$ be three points in $\mathbb{R}^3$ and let Π be the plane containing A, B and C.

   (a) Find the equation of the plane Π.

   (b) Suppose that

   $$L : \frac{x-1}{5} = \frac{y-1}{6} = z$$

   is a straight line passing through the point A and L' is the projection of L on Π.

   Find the equation of L'.

10. (a) Let Π be a plane in $\mathbb{R}^3$ given by the equation $Ax + By + Cz + D = 0$ and let $P(x_0, y_0, z_0)$ be a fixed point. Show that the perpendicular distance between Π and P is $\left| \frac{Ax_0 + By_0 + Cz_0 + D}{\sqrt{A^2 + B^2 + C^2}} \right|$.

    (b) Let $\Pi_1 : 2x - 2y + z - 4 = 0$ and $\Pi_2 : x + 2y - 2z = 0$ be two planes in $\mathbb{R}^3$.

    Find the equation of plane(s) passing through the intersection lines of plane bisecting the planes Π_1 and Π_2.

    (Hint: Suppose that $\mathbf{p}$ is a point lying on the required plane, then the distance between $\mathbf{p}$ and Π_1 equals to the distance between $\mathbf{p}$ and Π_2. Draw a picture to see why there are two such planes.)