# Tutorial 9

(Total) differentiability

Chain rule

Taylor's expansion of multivariable function

Implicit differentiation

$f^{-1}(c) \perp \nabla f(\vec{a})$

$D_{\vec{u}} f(\vec{a}) = \nabla f(\vec{a}) \cdot \vec{u}$

(try to think about the "meaning" of each ⤳)

① Let $f: \mathbb{R}^2 \to \mathbb{R}$ be a function (totally) differentiable at the point $(1,2)$. The (directional) derivative of $f$ at $(1,2)$ in the direction of $\hat{i}+\hat{j}$ is $2\sqrt{2}$ and that in the direction of $-2\hat{j}$ is $-3$. What is the (directional) derivative of $f$ in the direction of $-\hat{i}-2\hat{j}$?

Ans: Since $f$ is (totally) differentiable at $(1,2)$,

$$\nabla f(1,2) = \left(\frac{\partial f}{\partial x}(1,2), \frac{\partial f}{\partial y}(1,2)\right) \text{ exists.}$$

Let $x_0 := \frac{\partial f}{\partial x}(1,2)$ and $y_0 := \frac{\partial f}{\partial y}(1,2)$.

Again, by (total) differentiability of $f$, we have

$$D_{\frac{(1,1)}{\|(1,1)\|}} f(1,2) = \nabla f(1,2) \cdot \left(\frac{(1,1)}{\|(1,1)\|}\right)$$

and

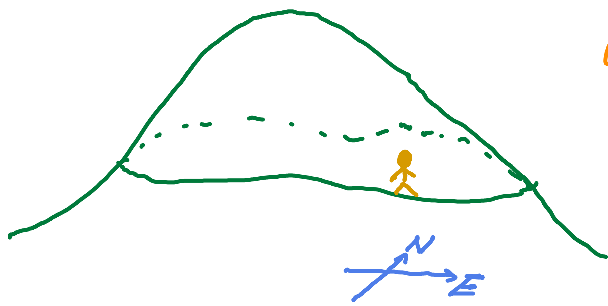
$$D_{\frac{(0,-2)}{\|(0,-2)\|}} f(1,2) = \nabla f(1,2) \cdot \left(\frac{(0,-2)}{\|(0,-2)\|}\right),$$

which gives

$$\begin{cases} 2\sqrt{2} = (x_0, y_0) \cdot \frac{(1,1)}{\sqrt{2}} \\ -3 = (x_0, y_0) \cdot \frac{(0,-2)}{2} \end{cases} \Leftrightarrow \begin{cases} 4 = x_0 + y_0 \\ -3 = -y_0 \end{cases}. \quad \therefore \nabla f(1,2) = (1,3)$$

$$\therefore D_{\frac{(-1,2)}{\|(-1,2)\|}} f(1,2) = (1,3) \cdot \frac{(-1,-2)}{\|(-1,-2)\|} = \frac{-1-6}{\sqrt{5}} = \frac{-7}{\sqrt{5}} \,/\!/$$

Remark for Q.1: Imagine you're walking on a hill whose surface can be modeled by a (totally) differentiable surface.




If you know the slopes in any two non-parallel directions, then you'll probably know the slope along all other directions.

② Consider the following equation:

$$xe^y + \sin(xy) + y - \ln 2 = 0$$

(a) Why can we talk about $\frac{dy}{dx}$ at a point $(0, \ln 2)$ ?

(b) Find $\frac{dy}{dx}$ at the point $(0, \ln 2)$.

Before discussing the answer, let me state two important results in advanced calculus.

$\boxed{\text{Inverse Function Theorem}}$: Suppose $f: \mathbb{R}^n \to \mathbb{R}^n$ is $C^1$ in an open set containing $a$, and $\det(Df(a)) \neq 0$.
Then there exist open $V \ni a$ and open $W \ni f(a)$ such that $f: V \to W$ has a continuous inverse $f^{-1}: W \to V$ which is differentiable and $\forall y \in W$, $D(f^{-1})(y) = Df(f^{-1}(y))^{-1}$.

$\boxed{\text{Implicit Function Theorem}}$: Suppose $f: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ is $C^1$ in an open set containing $(a, b)$ and $f(a, b) = 0$. Let $M$ be the $m \times m$ matrix $\left(\frac{\partial f_i}{\partial x_{n+j}}(a,b)\right)$. If $\det M \neq 0$, then there exist open $A \ni a$ and open $B \ni b$ such that $\forall x \in A$, $\exists!\, g(x) \in B$ such that $f(x, g(x)) = 0$ and $g$ is differentiable.

Ans: 2(a) Let $f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ be defined as

$$f(x,y) := xe^y + \sin(xy) + y - \ln(2).$$

Since $f$ is continuously differentiable on $\mathbb{R} \times \mathbb{R}$ and $f(0, \ln 2) = 0$, and

$$\frac{\partial f}{\partial y} = xe^y + x\cos(xy) + 1$$

is not zero at $(0, \ln 2)$, it follows from Implicit Function Theorem that there exist open sets $A$ and $B$ in $\mathbb{R}$ with $0 \in A$ and $\ln 2 \in B$ such that $y : A \longrightarrow B$ is a differentiable function.

Therefore, $\frac{dy}{dx}$ exists on $A$.

2(b) From (a), we may define a function $h : A \to \mathbb{R}$ by

$$h(x) = xe^{y(x)} + \sin(x \cdot y(x)) + y(x) - \ln(2),$$

such that $h(x) \equiv 0$.

Applying Chain Rule,

$$\begin{aligned} 0 &= \frac{dh}{dx} \\ &= \left[e^y + x e^y \frac{dy}{dx}\right] + \cos(x \cdot y) \cdot \left[x \frac{dy}{dx} + y\right] + \frac{dy}{dx} \\ &= (xe^y + x\cos(xy) + 1)\frac{dy}{dx} + y\cos(xy) + e^y \end{aligned}$$

$$\therefore \left.\frac{dy}{dx}\right|_{(x,y)=(0,\ln 2)} = \frac{-\ln 2 - 2}{1} = -(2 + \ln 2)$$

③ Recall the following thm in Week 9 lecture notes:

Let $\Omega$ be an open subset of $\mathbb{R}^n$. Let $f: \Omega \to \mathbb{R}$ be a function. Let $c \in \mathbb{R}$. Let $S := f^{-1}(c)$ and $a \in S$. Suppose $f$ is differentiable at $a$, and $\nabla f(a) \neq 0$. Then $\nabla f(a) \perp S$ at $a$.

(a) Is $S := f^{-1}(c)$ a surface?
(Actually how is a "surface" defined?)

(b) Let $f: \mathbb{R}^3 \longrightarrow \mathbb{R}$ be defined as $f(x,y,z) = 2z - x^2$.
Let $P_0 := (2,0,2)$.

(i) Show that $f^{-1}(0)$ is a "surface".

(ii) Find equation of tangent plane to the surface at $P_0$.

Ideas: 3(a) Let me try to rephrase a lemma on P.11 of Milnor's Topology from the Differentiable Viewpoint:

If $f: \mathbb{R}^m \longrightarrow \mathbb{R}$ is smooth and if $y \in \mathbb{R}$ is a regular value, (i.e. $\forall x \in f^{-1}(y)$, $Df(x) \neq \vec{0}$) then the set $f^{-1}(y) \subseteq \mathbb{R}^m$ is a smooth manifold ("surface") of dimension $(m-1)$. (Pf: use Inverse Function Theorem)

3(b)(i) $f$ is a polynomial function, which is smooth.
Let $(x,y,z) \in f^{-1}(0)$. $Df(x,y,z) = (-2x, 0, 2) \neq \vec{0}$.
By the above lemma, $f^{-1}(0)$ is indeed a "surface".

(ii) $\nabla f(2,0,2) = Df(2,0,2) = (-4, 0, 2)$
Since $f$ is differentiable at $(2,0,2)$ and $\nabla f(2,0,2) \neq 0$,
we have $\nabla f(2,0,2) \perp f^{-1}(0)$. $\therefore -4(x-2) + 2(z-2) = 0$ //