Tutorial 7

* C^r-condition in Mixed partial derivative thm (i.e. Clairaut's thm)
* C^1-condition for (total) differentiability

① Let $f: \mathbb{R}^2 \to \mathbb{R}$ be a function defined by

$$f(x,y) = \begin{cases} xy\,\dfrac{x^2-y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

(a) Show that $f_{xy}(0,0) \neq f_{yx}(0,0)$.

(b) What assumption in Clairaut's thm is not satisfied?

Ans: 1(a) Let's find the functions $f_x(x,y)$ and $f_y(x,y)$ first.

Suppose $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$.

$$f_x(x,y) = \frac{\partial}{\partial x}\left(\frac{y(x^3-y^2x)}{x^2+y^2}\right)$$

$$= \frac{(x^2+y^2)[y(3x^2-y^2)] - y(x^3-y^2x)(2x)}{(x^2+y^2)^2}$$

$$= \frac{y(x^4+4x^2y^2-y^4)}{(x^2+y^2)^2}.$$

Similarly, we have

$$f_y(x,y) = \frac{-x(y^4+4y^2x^2-x^4)}{(y^2+x^2)^2}.$$

$$f_x(0,0) = \lim_{t\to 0} \frac{f(0+t,0)-f(0,0)}{t} = \lim_{t\to 0} \frac{0-0}{t} = 0$$

$$f_y(0,0) = \lim_{t\to 0} \frac{f(0,0+t)-f(0,0)}{t} = \lim_{t\to 0} \frac{0-0}{t} = 0$$

(continue) Thus, we have

$$f_x(x,y) = \begin{cases} \dfrac{y(x^4+4x^2y^2-y^4)}{(x^2+y^2)^2} & \text{if } (x,y)\neq(0,0) \\ 0 & \text{if } (x,y)=(0,0) \end{cases}$$

and $$f_y(x,y) = \begin{cases} \dfrac{-x(y^4+4y^2x^2-x^4)}{(y^2+x^2)^2} & \text{if } (x,y)\neq(0,0) \\ 0 & \text{if } (x,y)=(0,0) \end{cases}$$

$$f_{xy}(0,0) = \frac{\partial f_x}{\partial y}(0,0)$$
$$= \lim_{t\to 0} \frac{f_x(0,0+t)-f_x(0,0)}{t}$$
$$= \lim_{t\to 0} \frac{-t^5/t^4 - 0}{t} = \lim_{t\to 0}(-1) = -1$$

$$f_{yx}(0,0) = \frac{\partial f_y}{\partial x}(0,0)$$
$$= \lim_{t\to 0} \frac{f_y(0+t,0)-f_y(0,0)}{t}$$
$$= \lim_{t\to 0} \frac{t^5/t^4 - 0}{t} = \lim_{t\to 0}(1) = 1 \neq -1$$

$\therefore$ $f_{xy}(0,0) \neq f_{yx}(0,0)$

1 (b) f is not C^2. It suffices to show that some (mixed) partial derivative of f is not continuous on $\mathbb{R}^2$.

$$f_{xy}(x,y) = \begin{cases} \dfrac{x^6+9x^4y^2-9x^2y^4-y^6}{(x^2+y^2)^3} & \text{if } (x,y)\neq(0,0) \\ -1 & \text{if } (x,y)=(0,0) \end{cases}$$

$$\lim_{\substack{(x,y)\to(0,0) \\ \text{along } x=0}} f_{xy}(x,y) = -1 \neq 1 = \lim_{\substack{(x,y)\to(0,0) \\ \text{along } y=0}} f_{xy}(x,y)$$

$\therefore$ f_{xy} is not continuous at $(0,0)$.

② Let $f: \mathbb{R}^2 \to \mathbb{R}$ be a function defined by

$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^4} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}.$$

(a) Show that $f_x(0,0)$ and $f_y(0,0)$ exist.

(b) Show that for each unit vector $\vec{u} \in \mathbb{R}^2$, $D_{\vec{u}} f(0,0)$ exists.

(c) Show that f is not continuous at $(0,0)$.

(thus, f is not (totally) differentiable at $(0,0)$)

(d) Is f C^1 ?

Ans: 2(a)

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{\left(\frac{h\cdot 0^2}{h^2+0^4} - 0\right)}{h} = \lim_{h\to 0} \frac{0}{h} = 0$$

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,0+h) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{\left(\frac{0\cdot h^2}{0^2+h^4} - 0\right)}{h} = \lim_{h\to 0} \frac{0}{h} = 0$$

$f_x(0,0)$ and $f_y(0,0)$ exist and equal to 0.

(b) Let $\vec{u}$ be a unit vector in $\mathbb{R}^2$.

There exist $u_1, u_2 \in \mathbb{R}$ such that $\vec{u} = (u_1, u_2)$.

Observe that

$$\left.\begin{aligned} D_{\vec{u}} f(0,0) &= \lim_{t\to 0} \frac{f((0,0)+t(u_1,u_2)) - f(0,0)}{t} \\ &= \lim_{t\to 0} \frac{u_1 u_2^2}{u_1^2 + t^2 u_2^4} \end{aligned}\right\} (\star)$$

Let $g(t) := \dfrac{u_1 u_2^2}{u_1^2 + t^2 u_2^4}$.

(continue) Case 1: Suppose $u_1 = 0$.

Since $\|\vec{u}\| = 1$, $u_2 \neq 0$. $g(t) = 0$

$\lim_{t \to 0} g(t) = 0$.

Case 2: Suppose $u_1 \neq 0$.

Since g is continuous at $t=0$,

$$\lim_{t \to 0} g(t) = g(0) = \frac{u_1 u_2^2}{u_1^2} = \frac{u_2^2}{u_1}.$$

$\therefore$ In any case, $\lim_{t \to 0} g(t)$ exists

From (☆), we may conclude that

$D_{\vec{u}} f(0,0)$ exists.

2(c) $$\lim_{\substack{(x,y)\to(0,0) \\ \text{along } x=y^2}} f(x,y) = \lim_{y \to 0} \frac{y^4}{y^4+y^4} = \lim_{y\to 0} \frac{1}{2} = \frac{1}{2}$$

$$\lim_{\substack{(x,y)\to(0,0) \\ \text{along } x=0}} f(x,y) = \lim_{y \to 0} \frac{0 \cdot y^2}{0^2+y^4} = \lim_{y\to 0} 0 = 0 \neq \frac{1}{2}$$

$\therefore \lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

$\therefore$ f is not continuous at $(0,0)$.

(thus, not differentiable at $(0,0)$)

2(d) No. (If f were C^1, then f would be differentiable at $(0,0)$. Contradiction.)

(Check: $\lim_{(x,y)\to(0,0)} f_x(x,y)$ DNE

f_x is not continuous at $(0,0)$)