Tutorial 3

This tutorial focuses mainly on
- polar coordinates ($r \in [0, \infty)$; $\theta \in \mathbb{R}$)
- topological terminology

Some questions come from MATH 2010D Problem Set 3.

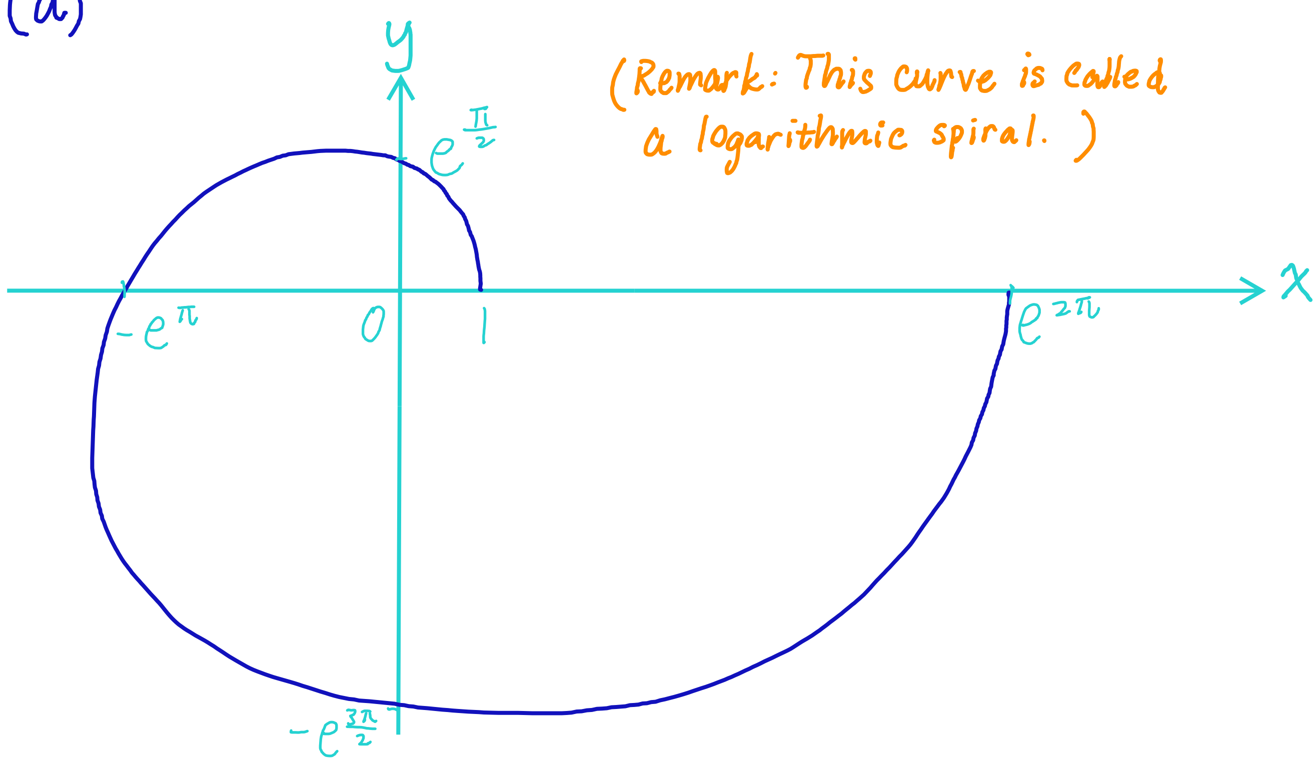
① (a) Sketch the polar curve $r = e^{\theta}$, where $\theta \in [0, 2\pi]$.

(b) Does the point $(r, \theta) = \left(e^{\frac{3\pi}{2}}, -\frac{\pi}{2}\right)$ lie in the curve in (a) ?

(c) Does the point in (b) satisfy the equation $r = e^{\theta}$?

(d) Find the arclength of the curve in (a).

Ans: (a)



(Remark: This curve is called a logarithmic spiral.)

(b) Yes.

(c) No.

(strange property about polar coordinates)

Ans: 1 (d)

Idea: Recall the definition of arclength of a smooth curve $\vec{\gamma}(t)$:

$$s = \int_a^b \|\vec{\gamma}\,'(t)\|\, dt.$$

If $\vec{\gamma}(t)$ is expressed as $\boxed{\vec{\gamma}(t) = \underbrace{(x(t), y(t))}_{\text{rectangular coordinates}}}$,

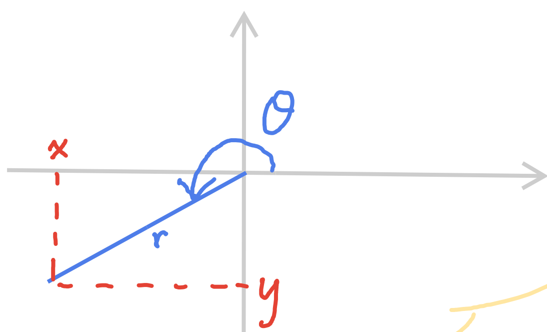
then we may compute $\|\vec{\gamma}\,'(t)\| = \sqrt{x'(t)^2 + y'(t)^2}$.

But now we are given a polar equation $r = e^\theta$, how do we get an expression such as the one in the green box?

length of angle arm

Recall: $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

rotation angle



our parameter

Let $\theta \in [0, 2\pi]$.

Define $\overbrace{r(\theta)}$

$$x(\theta) := e^\theta \cos\theta$$

$$y(\theta) := e^\theta \sin\theta$$

The curve in (a) is given by the parametrization

$$\vec{\gamma}(\theta) = \underbrace{(x(\theta), y(\theta))}_{\text{rectangular coordinates}}, \quad \forall \theta \in [0, 2\pi].$$

Therefore, arclength of the curve in (a)

$$= \int_0^{2\pi} \sqrt{x'(\theta)^2 + y'(\theta)^2}\, d\theta$$

$$= \int_0^{2\pi} e^\theta \cdot \sqrt{2}\, d\theta = \sqrt{2}\,(e^{2\pi} - 1).$$

② Let $S := \{\frac{1}{n} : n \in \mathbb{Z}^+\}$ be a subset of $\mathbb{R}$.

(a) Is S a closed subset of $\mathbb{R}$? Why?

(b) Write down $\mathrm{Int}(S)$ and ∂S. Explain.

Ans: (a) No.

Recall definitions (Week 3, P.8):
A subset $T \subseteq \mathbb{R}$ is called closed if $\mathbb{R} \setminus T$ is open.
A subset $U \subseteq \mathbb{R}$ is called open if
$\forall x \in U, \exists \delta > 0$ such that $B_\delta(x) \subseteq U$.
not open: $\exists x_0 \in U$ such that $\forall \delta > 0$, $B_\delta(x_0) \not\subseteq U$.

Take $x_0 := 0 \in \mathbb{R} \setminus S$. Let $\delta > 0$.
By Archimedean Property of $\mathbb{R}$, there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < \delta$. Thus, $\frac{1}{n} \in (-\delta, \delta)$.

$\therefore (-\delta, \delta) \not\subseteq \mathbb{R} \setminus S$.

$\therefore \mathbb{R} \setminus S$ is not open.

$\therefore S$ is not closed.

(b) Claim 1: $\mathrm{Int}(S) = \emptyset$.

Pf: Suppose it were true that $\mathrm{Int}(S) \neq \emptyset$.
There would exist $w \in \mathrm{Int}(S)$.
By def. of $\mathrm{Int}(S)$, $\exists \delta > 0$ such that $B_\delta(w) \subseteq S$.
Since $\exists$ irrational number $\iota \in (w-\delta, w+\delta)$ and $S \subseteq \mathbb{Q}$, we would have $B_\delta(w) \not\subseteq S$.
Contradiction would arise. Thus $\mathrm{Int}(S) = \emptyset$.

Claim 2: $\partial S = S \cup \{0\}$.

Outline of Pf:

$S \cup \{0\} \subseteq \partial S$: $\forall x \in S \cup \{0\}$, $\forall \delta > 0$,
$(x-\delta, x+\delta) \cap S \neq \emptyset$ and $(x-\delta, x+\delta) \cap (\mathbb{R} \setminus S) \neq \emptyset$.

$\partial S \subseteq S \cup \{0\}$: $\forall x \in \mathbb{R} \setminus (S \cup \{0\})$, $\exists \delta > 0$ such that
$(x-\delta, x+\delta) \subseteq \mathbb{R} \setminus S$ or $(x-\delta, x+\delta) \subseteq S$.

③ Let $S = \{(x,y) \in \mathbb{R}^2 : |x| \geq 1\}$ be a subset of $\mathbb{R}^2$. Show that S is not path connected.

Ans: Suppose S were a path connected set.

Since $(-1, 3)$ and $(1, 3)$ belong to S,

there would exist a continuous curve $\gamma : [0,1] \longrightarrow \mathbb{R}^2$

such that $\gamma(0) = (-1, 3)$ and $\gamma(1) = (1, 3)$,

and $\gamma([0,1]) \subseteq S$.

Define $(x(t), y(t)) := \gamma(t)$.

The component function $x : [0,1] \to \mathbb{R}$ would be continuous with $x(0) = -1$ and $x(1) = 1$.

Since $-1 < 0 < 1$, by Intermediate Value Theorem, there would exist $c \in (0,1)$ such that $x(c) = 0$.

Note that $\gamma(c)$ would be a point in S.

However, $\gamma(c) = (x(c), y(c)) = (0, y(c)) \notin S$.

Contradiction would arise.

Therefore S is not path connected.