Tutorial 11

Tools of $\boxed{\text{Finding Extrema}}$

- First derivative test
- Second derivative test
- Lagrange Multiplier(s)

① (a) State Second Derivative Test for multivariable function.

(b) Describe briefly how Second Derivative Test is obtained from Taylor's Theorem.

Ans: 1 (a) A "general version" of 2nd Derivative Test:

Let $n \in \mathbb{N} \setminus \{0\}$. Let Ω be an open subset of $\mathbb{R}^n$.

If $f : \Omega \longrightarrow \mathbb{R}$ is twice continuously differentiable,

and $\underline{a \in \Omega \text{ is a critical point of } f}$,

(i.e. $\nabla f(a) = \vec{0}$ in this case)

then $\left(\left(\underline{\forall h \in \mathbb{R}^n \setminus \{\vec{0}\},\ h^T (Hf(a)) h > 0} \right) \Rightarrow f(a) \text{ is a local min.} \right)$

(i.e. $Hf(a)$ is +ve definite)

and $\left(\left(\underline{\forall h \in \mathbb{R}^n \setminus \{\vec{0}\},\ h^T (Hf(a)) h < 0} \right) \Rightarrow f(a) \text{ is a local max.} \right)$

(i.e. $Hf(a)$ is -ve definite)

and $\Big(\underline{\exists\, h, k \in \mathbb{R}^n \setminus \{\vec{0}\} \text{ s.t. } h^T (Hf(a)) h > 0 \text{ and } k^T (Hf(a)) k < 0}$

(i.e. $Hf(a)$ is indefinite)

$\Rightarrow a$ is a saddle point of $f \Big)$

Remark: Using results from Linear Algebra (MATH 2040), (Spectral Theorem)
we may show that

$$\begin{pmatrix} Hf(a) \text{ is} \\ \text{positive definite} \end{pmatrix} \Leftrightarrow \begin{pmatrix} \text{all eigenvalue(s) of } Hf(a) \\ \text{is/are positive} \end{pmatrix} \Leftrightarrow \begin{pmatrix} \forall k \in \{1, \dots, n\}, \\ \det(H_k) > 0 \end{pmatrix}$$

$$\begin{pmatrix} Hf(a) \text{ is} \\ \text{negative definite} \end{pmatrix} \Leftrightarrow \begin{pmatrix} \text{all eigenvalue(s) of } Hf(a) \\ \text{is/are negative} \end{pmatrix} \Leftrightarrow \left(\det(H_k) \begin{cases} < 0 \text{ if } k \text{ is odd} \\ > 0 \text{ if } k \text{ is even} \end{cases} \right)$$

$$\begin{pmatrix} Hf(a) \text{ is} \\ \text{indefinite} \end{pmatrix} \Leftrightarrow \begin{pmatrix} \text{some eigenvalue of } Hf(a) > 0 \text{ and} \\ \text{some eigenvalue of } Hf(a) < 0 \end{pmatrix}$$

where H_k is the $k \times k$ submatrix of $Hf(a)$

defined as
$$H_k := \begin{pmatrix} f_{x_1x_1}(a) & f_{x_1x_2}(a) & \cdots & f_{x_1x_k}(a) \\ f_{x_2x_1}(a) & f_{x_2x_2}(a) & \cdots & \vdots \\ \vdots & & \ddots & \vdots \\ f_{x_kx_1}(a) & \cdots & \cdots & f_{x_kx_k}(a) \end{pmatrix}$$

The above $\Leftrightarrow$'s may provide easier ways of checking definiteness of $Hf(a)$.

By completing square or using the above remark, we may obtain a "less general" version of 2nd Derivative Test:

If $\Omega \subseteq \mathbb{R}^2$ is open, $f : \Omega \to \mathbb{R}$ is C^2, $a \in \Omega$ and $\nabla f(a) = \vec{0}$,
then $\left(\left(f_{xx}(a) f_{yy}(a) - f_{xy}(a)^2 > 0 \text{ and } f_{xx}(a) > 0 \right) \Rightarrow f(a) \text{ is a local min.} \right)$

and $\left(\left(f_{xx}(a) f_{yy}(a) - f_{xy}(a)^2 > 0 \text{ and } f_{xx}(a) < 0 \right) \Rightarrow f(a) \text{ is a local max.} \right)$

and $\left(\left(f_{xx}(a) f_{yy}(a) - f_{xy}(a)^2 < 0 \right) \Rightarrow a \text{ is a saddle point} \right)$

1 (b) (How to get 2nd Derivative Test from Taylor's Thm?)

Rough idea (not rigorously written):

Taylor's Thm tells us when x is "close to" a,

$$f(x) \approx f(a) + \underbrace{\cancel{\nabla f(a)}}_{0} \cdot (x-a) + \frac{1}{2!}(x-a)^T Hf(a)(x-a),$$

0 (by assumption in 2nd Derivative Test)

which basically gives you the "general version" of 2nd Derivative Test.

We could continue to "show" the "less general version": (by completing square)

$$f(x) - f(a) \approx \frac{1}{2!}(x-a)^T Hf(a)(x-a)$$

$f: \Omega \to \mathbb{R}$, $\Omega \subseteq \mathbb{R}^2$

$$=: \frac{1}{2!}(h_1, h_2)\begin{pmatrix} f_{xx}(a) & f_{xy}(a) \\ f_{yx}(a) & f_{yy}(a) \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$$

$$=: \frac{1}{2!}(h_1, h_2)\begin{pmatrix} A & B \\ B & C \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$$

$$= \frac{1}{2!}(h_1, h_2)\begin{pmatrix} Ah_1 + Bh_2 \\ Bh_1 + Ch_2 \end{pmatrix}$$

$$= \frac{1}{2!}\Big((Ah_1 + Bh_2)h_1 + (Bh_1 + Ch_2)h_2 \Big)$$

$$= \frac{1}{2!}\Big(A(h_1)^2 + 2B(h_1)(h_2) + C(h_2)^2 \Big)$$

"WLOG" assume $A \neq 0$

$$= \frac{1}{2!} A\left[(h_1)^2 + \left(\frac{2B}{A}h_2\right)(h_1) + \left(\frac{C}{A}h_2^2\right) \right]$$

$$= \frac{1}{2!} A\left[\left(h_1 + \frac{B}{A}h_2\right)^2 - \left(\frac{B}{A}h_2\right)^2 + \left(\frac{C}{A}h_2^2\right) \right]$$

$$\therefore f(x) - f(a) \approx \frac{1}{2!} A\left[\left(h_1 + \frac{B}{A}h_2\right)^2 + \frac{AC - B^2}{A^2}h_2^2 \right],$$

which basically gives you the "less general" version.

② Define $g: B_{\frac{1}{2}}(1,1,1) \subseteq \mathbb{R}^3 \longrightarrow \mathbb{R}$ by

$$g(x,y,z) = -\frac{1}{4}\left(x^{-4}+y^{-4}+z^{-4}\right)+yz-x-2y-2z.$$

Let $a := (1,1,1)$. Determine whether $a=(1,1,1)$ corresponds to a local minimum, local maximum, or saddle point.

Ans: Let $(x,y,z) \in B_{\frac{1}{2}}(1,1,1)$.

$$\frac{\partial g}{\partial x}(x,y,z) = x^{-5}-1. \qquad \frac{\partial g}{\partial y}(x,y,z) = y^{-5}+z-2. \qquad \frac{\partial g}{\partial z}(x,y,z) = z^{-5}+y-2.$$

$$\therefore \nabla g(x,y,z) = \left(g_x(x,y,z),\ g_y(x,y,z),\ g_z(x,y,z)\right)$$

$$= \left(x^{-5}-1,\ y^{-5}+z-2,\ z^{-5}+y-2\right)$$

Since $B_{\frac{1}{2}}(1,1,1)$ is open, g is C^2 and

$$\nabla g(1,1,1) = (0,0,0),$$

We may use 2nd Derivative Test (general version).

$$Hg(x,y,z) = \begin{pmatrix} g_{xx}(x,y,z) & g_{xy}(x,y,z) & g_{xz}(x,y,z) \\ g_{yx}(x,y,z) & g_{yy}(x,y,z) & g_{yz}(x,y,z) \\ g_{zx}(x,y,z) & g_{zy}(x,y,z) & g_{zz}(x,y,z) \end{pmatrix}$$

$$= \begin{pmatrix} -5x^{-6} & 0 & 0 \\ 0 & -5y^{-6} & 1 \\ 0 & 1 & -5z^{-6} \end{pmatrix}$$

$$Hg(1,1,1) = \begin{pmatrix} -5 & 0 & 0 \\ 0 & -5 & 1 \\ 0 & 1 & -5 \end{pmatrix}$$

Using the notation in remark for 1 (a), we have

$$\det(H_1) = \det((-5)) = -5 < 0$$

$$\det(H_2) = \det\left(\begin{pmatrix} -5 & 0 \\ 0 & -5 \end{pmatrix}\right) = 25 > 0$$

$$\det(H_3) = \det\left(\begin{pmatrix} -5 & 0 & 0 \\ 0 & -5 & 1 \\ 0 & 1 & -5 \end{pmatrix}\right)$$

$$= (-5)^3 - (-5) = (-5)\cdot 24 < 0$$

$\therefore$ $Hg(1,1,1)$ is negative definite.

$\therefore$ By 2nd Derivative Test,

g attains local maximum at $a = (1,1,1)$.