Topics covered in Lecture 3

(partial)

Two examples on special limits

1. $\lim_{x\to 0}\frac{1-\cos x}{x\sin x} = \lim_{x\to 0}\frac{2\sin^2\left(\frac{x}{2}\right)}{x\sin x}$ by the double angle formula $\cos x = 1-\sin^2\left(\frac{x}{2}\right)$

$$= \lim_{x\to 0}\frac{2\left(\frac{x}{2}\right)^2\sin^2\left(\frac{x}{2}\right)}{\left(\frac{x}{2}\right)^2 x\sin x} == \lim_{x\to 0}\frac{2\left(\frac{x}{2}\right)^2\sin^2\left(\frac{x}{2}\right)}{\left(\frac{x}{2}\right)^2 x\sin x} = \lim_{x\to 0}\frac{2\left(\frac{x}{4}\right)}{\sin x} = \frac{1}{2}$$

2. $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = ?$

Ans: Let $a_n = \left(1+\frac{1}{n}\right)^n, n = 1,2,3,\cdots$ We will use (i) the sequence $\{a_n\}$ is increasing (we usually write sequence in this way, enclosing it by { and } on the left and on the right) and (ii) it is bounded above. These two facts would show that the sequence has a limit.

Proof of (i) $a_n < a_{n+1}$ (i.e. $\left(1+\frac{1}{n}\right)^n < \left(1+\frac{1}{n+1}\right)^{n+1}$)

- To show this, we need to use (a) Mathematical Induction and the (b) Arithmetic Mean – Geometric Mean inequality. It says if $b, b_2, \cdots$ non-negative numbers, then

$$(b_1 b_2 \cdots b_n)^{\frac{1}{n}} \le \frac{b_1+b_2+\cdots+b_n}{n}$$

- **Remark** We will omit this, because it's technical

Proof of (ii) This is more elementary. One uses the Binomial Theorem.

$$\left(1+\frac{1}{n}\right)^n = 1 + n\left(\frac{1}{n}\right) + \frac{n(n-1)}{2!}\frac{1}{n^2} + \cdots +$$

$$+\frac{n(n-1)\cdots(n-k+1)}{k!}\frac{1}{n^k} + \cdots + +\frac{n(n-1)\cdots(n-n+1)}{n!}\frac{1}{n^n}$$

$$= 1 + n\left(\frac{1}{n}\right) + \frac{n(n-1)}{n^2}\frac{1}{2!} + \cdots +$$

$$+\frac{n(n-1)\cdots(n-k+1)}{n^k}\frac{1}{k!} + \cdots + +\frac{n(n-1)\cdots(n-n+1)}{n^n}\frac{1}{n!}$$

$$= 1 + 1 + \frac{n(n-1)}{n\cdot n}\frac{1}{2!} + \cdots +$$

$$+\frac{n(n-1)\cdots(n-k+1)}{n\cdot n\cdot\cdots\cdot n}\frac{1}{k!}+\cdots++\frac{n(n-1)\cdots(n-n+1)}{n\cdot n\cdot\cdots\cdots\cdot n}\frac{1}{n!}$$

As each of the terms $\frac{n}{n},\frac{n-1}{n},\cdots,\frac{n-k+1}{n},\cdots$ are ≤ 1, we have

$$\left(1+\frac{1}{n}\right)^n \leq 1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}\cdots+\frac{1}{n!}$$

$$\leq 1+1+\frac{1}{2\cdot 1}+\frac{1}{3\cdot 2}+\frac{1}{4\cdot 3}\cdots+\frac{1}{n\cdot(n-1)}$$

$$=1+1+\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)\cdots+\left(\frac{1}{n-1}-\frac{1}{n}\right)\leq 1+1+1-\frac{1}{n}$$

$$\leq 3$$

Hence we have found a number 3 which is an upper bound for the sequence $\left(1+\frac{1}{n}\right)^n$.

- By using these facts, we get $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \text{a number less than } 3.$

  This limiting number is given the symbol e.
- One can show that $e \approx 2.71828$

Remark

Take a look at this webpage it you want to know more:
https://courses.lumenlearning.com/boundless-algebra/chapter/the-real-number-e/