First of all, let's review CW3.

Two questions worth mentioning:

1) $f \circ g$, f cont. g cont. given by some rules.

Show $f \circ g$ cont. at some specific point.

2) $f(x+y) = f(x)f(y)$. If f is diff. at 0, show f is diff. at x

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Lecture 5

(part 2)

Done: proved Rolle's Theorem.

Ideas to prove Rolle's Theorem.

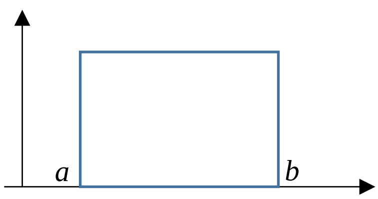
Main ideas recap/ kick start by using Extreme Value theorem/ then 2 cases can occur, either one of the global max/min point is "inside" (a, b) or both on boundary of this interval (i.e. one of them is the point a, the other the point b).

In the first case

we use Fermat's Lemma

In this case, one of the global max/min point (also known as "extreme" point) is "inside" (a, b), so it looks like either the red point on the left or on the right. Either way, its derivative is zero there (by Fermat's Lemma).

Case 2: It's a horizontal line. So $f'(x) = 0 \; \forall x$ in (a, b)



What Use do we have for RT?

RT implies CMVT, and CMVT implies LMVT

Application of each of them.

LMVT

We've done $|\sin x - \sin y| \leq |x - y|$

Better Applications

Question. If f satisfies $f'(x) > 0$ for each x in (a, b), show that f is a strictly increasing function.

Proof

Consider $f(x_2) - f(x_1)$.

Goal: Show that this expression satisfies > 0.

Details: $\frac{f(x_2)-f(x_1)}{x_2-x_1} = f'(\xi)$ for some ξ between x_1 *and* x_2

(i.e. $\exists\xi$ between x_1 and x_2) ("for some" has the shorthand symbol $\exists$)

But $f'(x) > 0 \ \ \forall x$

Therefore $\frac{f(x_2)-f(x_1)}{x_2-x_1} = f'(\xi) > 0$ (the term $f'(\xi) > 0$, because $f'(x), \forall x$)

Now the fraction $\frac{f(x_2)-f(x_1)}{x_2-x_1}$ has a positive denominator, because $x_2 - x_1 > 0$, so $f(x_2) - f(x_1) > 0$ also.

Question. Suppose f is differentiable (i.e. $f'(x)$ exists) for each real number and also $f'(x) = constant$, then $f(x) = ax + b$.

Proof. Many methods, one of them is by means of the MVTs (the "mean value theorem(s)")

You can try a simpler version of this result by using Rolle's Theorem. The question is:

Question: Suppose f is differentiable (i.e. $f'(x)$ exists) for each real number and also $f'(x) = 0, \forall x$. Then $f(x) = K \ \exists K$

L'H Rule

Browse through this:

https://www.math.ucdavis.edu/~kouba/CalcOneDIRECTORY/lhopitaldirectory/LHopital.html

L'Hopital Rule is useful, but how do we get it?

Answer: By CMVT.

Two Simplest Cases of L'Hopital Rule

Suppose that $\lim_{x\to a} f(x) = 0, \lim_{x\to a} g(x) = 0$, and that functions f and g are differentiable on an open interval I containing a. Assume also that $g'(x) \neq 0$ in I if $x \neq a$. Then

$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$ as long as the limit on the right-hand side is finite.

To illustrate the idea, we will prove an even more simple case, assuming that $f(a) = g(a) = 0$.

More precisely, it is:

Suppose that $f(a) = 0, g(a) = 0$, and that functions f and g are differentiable on an open interval I containing a. Assume also that $g'(x) \neq 0$ in I if $x \neq a$. Then

$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$ as long as the limit on the right-hand side is finite.

(Source: https://web.ma.utexas.edu/users/sadun/F11/408N/L%27Hopital.pdf)

Proof: consider $(f(x) - f(a))/(g(x) - g(a)) = f(x)/g(x)$

applying CMVT to the LHS, we get

$$f'(\xi)/g'(\xi) \; = \; (f(x) - f(a))/(g(x) - g(a))$$

Letting $x \to \; a$ gives (coz ξ bet' x and a), $\xi \to a$

Since $\lim_{x \to a} f'(\xi) \; = \; \lim_{\xi \to a} f'(\xi)$, so done.