Lecture 1-2

Calculus is a study of “functions”.

A function is basically a “rule” of the following form:

$$f: x \mapsto y$$

(Here the symbol f is the “name” of the function. E.g. $\sin: x \rightarrow y$ would be sine function. $\exp: x \mapsto y$ would be the exponential function).

Remark

Sometimes, we write the symbol over the “arrow with a vertical line”.

The “totality” or “collection” or “set” of all such x is called the domain (or “maximal” or “natural” domain) of the function.

Because the y here depends on what x we choose, we call x the “independent variable” and y the dependent variable.

The fact that y is the “result” of applying the rule f to the independent variable leads the writing $y = f(x)$. It means “y is the result of (evaluating) the function at x”.

Important Point

Each time when you input an x, there is one and only one answer for y.

Counterexample

The rule $x \mapsto \pm\sqrt{x}$ is not a function (because it has two values each time).

Examples of Domains

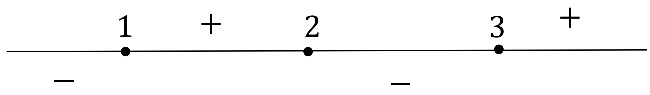
Consider the function given by $y = \frac{x-1}{(x-2)(x-3)}$. Its domain is the set (in interval notation) $(-\infty, 2) \cup (2,3) \cup (3, \infty)$.

Picture (or “graph”) of the function

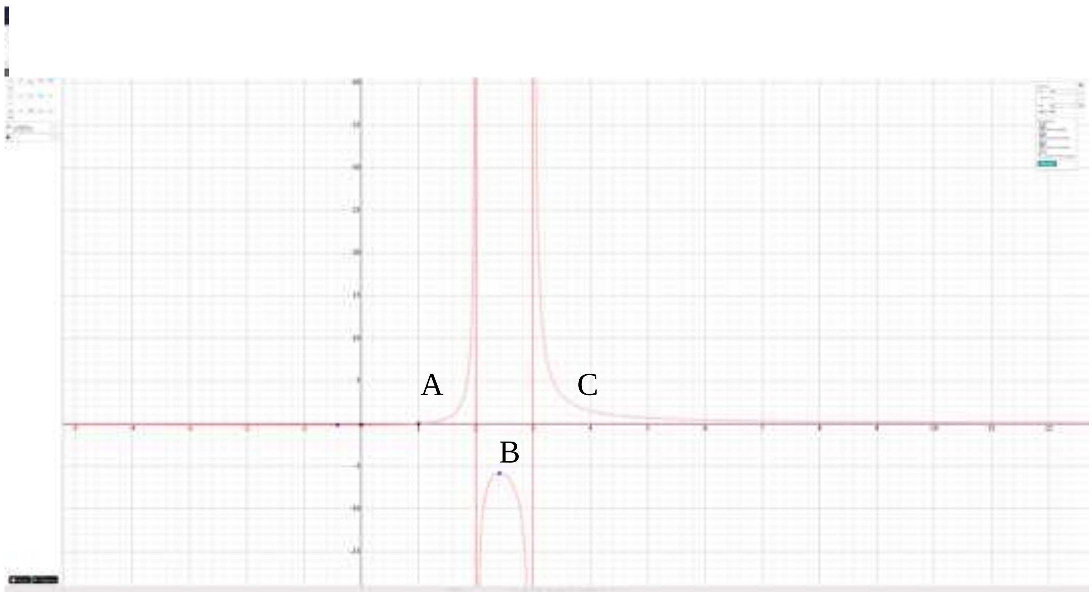
One can quickly sketch a rough graph of it.

Method:

1. Consider the three numbers $1,2$ and 3, check their order;
2. Note that $1 < 2 < 3$;
3. Check which of these 3 numbers is a "zero" of the "numerator". Which of them are "zeros" of the denominator;
4. Since $x < 1$ implies $x < 2$ and $x < 3$, we have



5. Also, note that the curve (described by the function) passes through the point $(1,0)$, because 1 is a zero of the function;
6. Next, we try to study what happens (i) when $x \to -\infty, x \to +\infty, x \to 1^-, x \to 1^+, x \to 2^-, x \to 2^+, x \to 3^-$ and $x \to 2^+$.
7. Having done all the above, we get the picture similar to the following one:



This is one reason why we have to study the concept of "limits".

Remarks on the picture

1. Our method is very crude, it doesn't tell us whether there is no/one local minimum or more local minima in region A. Same for region C.
2. Similarly, it doesn't tell us how many local maxima or local minima there are in region B.

More on limit concept

Notation

We use the notation $\lim_{x\to c} f(x) = L$ to mean "when x goes nearer and nearer to the point c, the function goes nearer and nearer to L.

Most of the time, c and L are finite numbers.

In this case, in order that $\lim_{x\to c} f(x) = L$ is true, we need to ensure that

1. $\lim_{x\to c^-} f(x) = L_1$ holds;
2. $\lim_{x\to c^+} f(x) = L_2$ holds;
3. The numbers L_1 is equal to the number L_2.

That is, the left-hand limit is equal to the right-hand limit.

Remark

If these two numbers are not equal, or one (or both) of them does/do not exist, then we say $\lim_{x\to c} f(x)$ doesn't exist.

<u>The case when $c = -\infty, +\infty$ or $L = -\infty, +\infty$</u>

Previously, we concentrated on the case when both c and L are finite numbers. For simplicity!

Case 1) $c = +\infty$ (case for $-\infty$ is similar!)

In this case, only left-hand limit exists (why? Because we cannot approach $+\infty$ from the right-hand side);

Case 2) c is a finite number and $L = +\infty$ (case for $-\infty$ is similar!)

In this case, $\lim_{x\to c} f(x) = L = +\infty$ and we say "the limit of f when $x \to c$ (meaning "x goes nearer and nearer to c") cannot exist.

Example

$$f(x) = 1/|x|$$

Then when $x \to 0^-$, $f(x) \to +\infty$.
Same thing happens when $x \to c^+$ (i.e. $f(x) \to +\infty$).

But "infinity" is not a number (it's just a concept, meaning the function's values become larger and larger beyond any bound!)

So we say "$\lim_{x\to c} f(x)$ doesn't exist".

An Important Kind of Limit

The concept of limit originated probably the time when people began to think about the concept of "differentiation".

That is, one is interested in limits of the form:

$$\lim_{x\to 0} \frac{f(c+x)-f(c)}{x}$$

The fraction (or "quotient") $\frac{f(c+x)-f(c)}{x}$ is obviously "not defined" when $x = 0$. But we can still study its "limiting behavior" when $x \to 0$.

Example 1

Suppose $f(u) = u^2$. Compute the limit

$$\lim_{x\to 0} \frac{f(c+x)-f(c)}{x}$$

Ans:

$$\frac{f(c+x)-f(c)}{x} = \frac{x^2+2xc+c^2-c^2}{x}$$

After simplification, we obtain $\frac{x^2+2xc}{x} = x + 2c$

(cancellation is allowed, because $x \neq 0$).

Finally, we let $x \to 0$ and get

$$\lim_{x\to 0} \frac{f(c+x)-f(c)}{x} = \lim_{x\to 0}(x+2c) = 2c.$$

Remark

The above limit is the justification of what we learned in school, i.e. $\frac{dx^2}{dx} = 2x$ calculated at the point $x = c$.

Example 2

Suppose $f(u) = u^n$, n is a natural number. Compute the limit

$$\lim_{x\to 0} \frac{f(c+x) - f(c)}{x}$$

Ans: First we study the fraction $\frac{f(c+x)-f(c)}{x} = \frac{(c+x)^n - c^n}{x}$

But $(c+x)^n = (c+x)(c+x)\cdots(c+x)$ multiplied n times.

Here we notice that the right-hand side is of the form

$$c^n + nc^{n-1}x + \text{polynomial starting at } x^2$$

,

Therefore, $\frac{(c+x)^n - c^n}{x} = \frac{c^n + nc^{n-1}x + \text{sum involving } x^2, x^3, \cdots, x^n - c^n}{x}$

$$= nc^{n-1} + \text{sum involving } x, x^2, \cdots, x^{n-1}$$

Finally we let $x \to 0$ and obtain

$$\lim_{x\to 0} \frac{(c+x)^n - c^n}{x} = \lim_{x\to 0}(nc^{n-1} + \text{sum involving } x, x^2, \cdots, x^{n-1})$$

$$= nc^{n-1}$$

which is the proof of the formula $\frac{dx^n}{dx} = nx^{n-1}$ calculated at the point $x = c$.

Remark

n is a natural number here (i.e. numbers beginning from 0). In the case when n is any real no., we also have a similar formula, but the proof is different (we use Chain Rule).

First Examples of Functions

As first examples, we have the functions such as

- Polynomials, i.e. functions like $x^4 + 100x + 1$. A polynomial is a finite sum involving powers of x and numbers.

- sine, cosine functions; e^x function (or "exponential function"), $\ln y$ function (or log function).

  Domain
- For polynomials, the domain is $(-\infty, +\infty)$, because each time we choose a value for x, the polynomial can be computed.
- For sine, cosine, exponential functions the domains are also $(-\infty, +\infty)$, if you use the right-angled triangle definitions for them.
- For log function, the domain is $(0, +\infty)$.

More Examples of Functions
- (Rational function) A rational function is a quotient of two polynomials, e.g. $\frac{x^2+3x+2}{x^3-1}$.
- (Tangent function) $\tan x$ is the quotient $\frac{\sin x}{\cos x}$.

Domains of rational function & tangent function

The domains are $(-\infty, +\infty)\backslash\{\text{zeros of denominator}\}$

E.g. Domain of $\frac{x^2+3x+2}{x^3-1}$ is the set $(-\infty, +\infty)\backslash\{1\}$. In interval notation, it is then the set $(-\infty, 1) \cup (1, +\infty)$.

Domain of $\tan x$ is the set

$(-\infty, +\infty)\backslash\{\cdots, -\pi - \frac{\pi}{2}, -\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi + \frac{\pi}{2}, \cdots\}$

Arithmetic of Limits

Suppose c, L are

(i) two finite numbers

(ii) $\lim_{x \to c} f(x) = L$ exists (i.e. left-hand limit = right-hand limit);

(iii) $\lim_{x \to c} g(x) = M$.

Conclusion: We have $\lim_{x \to c} f(x) * g(x) = \lim_{x \to c} f(x) * \lim_{x \to c} g(x)$,

where * means $+, -, \times$ or $\div$. In the case of $\div$, one has to further assume that $\lim_{x \to c} g(x) \neq 0$.

Remark

As before, we have similar arithmetic rules in the cases when c, L are infinities. But there, one has to be more careful.

Examples of Limit Computations

It is useful to read the pages in the webpage

http://www.intuitive-calculus.com/solving-limits.html

Two Useful Theorems for Finding Limits

1. Assumptions: Let $f(x)$ satisfy

(i) (bounded above) That is $f(x) \leq C$ for all x

(ii) (increasing) That is $f(x) \leq f(y)$ whenever $x < y$

Conclusion: $\lim_{x \to \infty} f(x) = L$ exists

Remarks

- Similar result holds if we have "bounded below" & "decreasing".
- One can also replace ∞ by any finite number a. The conclusion then becomes: $\lim_{x \to a} f(x) = L$.

- (Special Case of Sequence)

The above result is also true if the function's domain is the set of natural numbers (starting from one). i.e. $a: n \mapsto y = a(n)$

Let $a(n)$ satisfy

(iii) (bounded above) That is $a(n) \leq C$ for all $n = 1,2,3, \cdots$

(iv) (increasing) That is $a(n) \leq a(m)$ whenever $n < m$

Then $\lim_{x \to \infty} a(n) = L$ exists

Remark

Usually, we use the notation a_n for $a(n)$.

Sandwich/Squeeze Theorem

This theorem says the following:

- Let $h(x), f(x), g(x)$ be 3 functions, satisfying $h(x) \leq f(x) \leq g(x)$, $a < x < b$ and $x \neq c$ (c is some point between a and b).

- Furthermore, suppose $\lim_{x \to c} h(x) = L$ and $\lim_{x \to c} g(x) = L$.
- Then, the middle function, i.e. $f(x)$ also has a limit when $x \to c$ and the limit is also L. Mathematically, this is written as $\lim_{x \to c} f(x) = L$.

Examples

1. Let $f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & , x \neq 0 \\ \text{undefined} & , x = 0 \end{cases}$

   Find $\lim_{x \to 0} f(x)$.

Ans: Use Sandwich Theorem.
From the picture, one sees that the curve $y = f(x)$ is sandwiched between an upper V-shape and a lower V-shape. To say this more precisely, one can consider the absolute value of $f(x)$, i.e. the function $g(x) = |f(x)|$.

Using the fact that sine, cosine functions are always between -1 and $+1$, we obtain

$$0 \leq |f(x)| = |x \sin\left(\frac{1}{x}\right)| \leq |x|$$

So the middle function, i.e. $|f(x)|$ is sandwiched between the zero-function and the absolute value function.

As $\lim_{x \to 0} 0 = 0$ and $\lim_{x \to 0} |x| = 0$, by Sandwich Theorem, the middle function also has zero limit, i.e. $\lim_{x \to 0} |f(x)| = 0$.

Next, as $-|f(x)| \leq f(x) \leq |f(x)|$, $x \neq c$, therefore by Sandwich Theorem again, we have

$$-\lim_{x \to 0} |f(x)| \leq \lim_{x \to 0} f(x) \leq \lim_{x \to 0} |f(x)|$$

This shows that the “middle” function also has zero limit.

Conclusion: $\lim_{x \to 0} f(x) = 0$.

2. Let $f(x) = \begin{cases} x^{2n} \sin\left(\frac{1}{x}\right) & , \ x \neq 0 \\ 0 & , \ x = 0 \end{cases}$

Find $\lim_{x \to 0} \frac{f(x)-f(0)}{x-0}$. Here n is a natural number starting from 1.

Ans: $\frac{f(x)-f(0)}{x-0} = \frac{x^{2n} \sin\left(\frac{1}{x}\right)-0}{x} = x^{2n-1} \sin\left(\frac{1}{x}\right)$, where $n = 1,2,3,\cdots$ and $x \neq 0$.

Therefore $\left|\frac{f(x)-f(0)}{x-0}\right| = \left|x^{2n-1} \sin\left(\frac{1}{x}\right)\right|$, where $2n-1 = 1,3,5 \cdots$ and $x \neq 0$.

By Sandwich Theorem, $0 \leq \left|x^{2n-1} \sin\left(\frac{1}{x}\right)\right| \leq |x|^{2n-1}$.

Therefore, the middle function is sandwiched between the zero-function and the function $|x|^{2n-1}$. Since $\lim_{x \to 0} |x|^{2n-1} = 0$, the middle function also has zero limit.

3. Let $f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & , \ x \neq 0 \\ 0 & , \ x = 0 \end{cases}$

Find $\lim_{x \to 0} \frac{f(x)-f(0)}{x-0}$.

Ans: $\frac{f(x)-f(0)}{x-0} = \sin\left(\frac{1}{x}\right), x \neq 0$. One can show by choosing numbers like

$$x_1 = \frac{1}{\frac{\pi}{2}}, x_2 = \frac{1}{\frac{\pi}{2}+\pi}, x_3 = \frac{1}{\frac{\pi}{2}+2\pi}, \cdots, x_n = \frac{1}{\frac{\pi}{2}+(n-1)\pi}, \cdots$$

Therefore $\sin\left(\frac{1}{x_n}\right) = 1$ if n is an odd number, $\sin\left(\frac{1}{x_n}\right) = -1$, if n is an even number, while $x_n \to 0$. Since $\sin\left(\frac{1}{x}\right)$ oscilates as x_n goes to zero, the function doesn't have a limit as $x_n \to 0$.

Remark

Special Limits

Two special limits which we quite often use are the following:

- $\lim_{x \to 0} \frac{\sin x}{x} = 1$
- $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e$

Inverse Function

Given a function $y = f(x)$, one may "solve" x in terms of y and get x depending on y. If this can be done, then this "x in terms of y is an inverse function of f.

More mathematically, we can write $y = f(x)$ implies $x = g(y)$, where the function g performs the "reverse" of f. That is, $y = f(x) = f\big(g(y)\big) = y$.

Examples

1. $y = e^x$ implies $x = \ln(y)$ because $y = e^x = e^{\ln y} = y$

Question: What are the domains of e^x and of $\ln y$?

2. $y = x^2$, then $x = \sqrt{y}$ is the inverse function of x^2.

Domains: $(-\infty, \infty)$ and $(0, \infty)$.

Remark

In many textbooks, there is a graphical method to find inverse function.

Method:

1. Given a function $y = f(x)$. Draw the straight line $y = x$
2. Reflect the curve $y = f(x)$ about this line.
3. The reflected curve is the inverse function of f.

(see https://www.purplemath.com/modules/invrsfcn.htm)

Implicit Function

Sometimes, a function is not given by formulas like $y = f(x)$, but is hidden inside some equation relating x and y.

Example

$$x^2 + y^2 - 1 = 0$$

Then $y = \pm\sqrt{1 - x^2}$ and $x = \pm\sqrt{1 - y^2}$

So the equation $x^2 + y^2 - 1 = 0$ describes four functions. Two of them depende on the variable x, the other two depend on the variable y.

Remark

The equation is of the form $F(x, y) = 0$. On the left-hande side, we have a function depending on x and y. On the right-hand side, we have a zero (actually any constant number is O.K.)

There is a theorem, which says if we have an "equation" of the form

$F(x, y) = 0$, then it is always true that y is a function of x (written as $y = y(x)$) or x is a function of y (written $x = x(y)$)

Example

Lemniscate (see https://en.wikipedia.org/wiki/Lemniscate_of_Bernoulli)
function given by

$$(x^2 + y^2)^2 - x^2 + y^2 = 0$$

Question:
How would you write down y in terms of x?

Range

Given a function $y = f(x)$ and a domain D (just a notation!) we can ask "for which y is this equation solvable?" That is, does there exist x such that $y = f(x)$ has a solution?
x
The range of f is the set of all "solvable" y is the range of f, written as $R(f) = \{y | y = f(x) \text{ is solvable for some } x$ in the domain$\}$.

Example

Find the range of the function $f(x) = \frac{1}{1+x^2}$.

Ans:

Consider the equation $y = f(x) = \frac{1}{1+x^2}$

We want to ask "for which x is the equation solvable?"

Method 1 (intuitive method)

Since $0 \le \frac{1}{1+x^2} \le 1$, $f(0) = 1$ and the curve $y = \frac{1}{1+x^2}$ is continuous (i.e. unbroken). Also we have $\lim_{x \to -\infty} \frac{1}{1+x^2} = 0^+$ and $\lim_{x \to +\infty} \frac{1}{1+x^2} = 0^+$, therefore $R(f) = (0,1]$.

Method 2 (algebraic)

Consider the equation $y = \frac{1}{1+x^2}$ which is equivalent to $y(1 + x^2) - 1 = 0$.

This is a quadratic equation in x. (Think of y and $y - 1$ as "coefficients"!)

$$yx^2 + (y-1) = 0$$

Coefficient of x^2 is y, coeff. Of x is 0, "constant" term is $y-1$.

So using $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$, we get $b^2 - 4ac = 0^2 - 4y(y-1) = 4y(1-y)$

$b^2 - 4ac \geq 0$ if an only if $y \geq 0$ & $1 - y \geq 0$ which gives $1 \geq y \geq 0$.

One more requirement – the term $2a$ in the denominator has to be nonzero, so we need also $y \neq 0$.

Conclusion. $R(f) = (0,1]$.