1510 Lecture 6-2

(L6-1 preempted due to public holiday)

Keyword: Taylor's Theorem, its apps

The statement of TT:

$$f(x) = f(c) + f'(c)(x-c) + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-c)^{n+1}$$

Q: The above line is the "conclusion" of Taylor's Theorem. How about the assumptions ?

Answer: We need the following assumption:

- f is differentiable $n+1$ times in an open interval containing the point c.

Do-It-Yourself Question: Is this assumption enough?

Apps:

1) estimate e^1 with an error less than 0.01.

2) second derivative test (that said, we should also mention first deriv. test) (didn't mention yet!)

Some Names

Taylor's Polynomial (or Taylor Polynomial, or Truncated T. S.)

- If we have

$$f(x) = f(c) + f'(c)(x-c) + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-c)^{n+1}$$

and the error term satisfies $\lim_{n\to\infty}\left|\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-c)^{n+1}\right| = 0$, then we get

$$f(x) = f(c) + f'(c)(x-c) + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots$$

or (if we use summation notation)

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

The right-hand side (without error term!) is called the Taylor Series of the function f centered at the point c.

- If we "truncate" (i.e. "delete") the Taylor Series at the n^{th} term, then we get a Taylor Polynomial of order n, centered at the point c. The highest power term is $(x-c)^n$, i.e.

$$f(c) + f'(c)(x-c) + \cdots + \frac{f^{(n)}(c)}{(n+1)!}(x-c)^{n+1}$$

or

$$\sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

Sometimes we use the notation $TP_n(x,c)$ to mean "Taylor Polynomial of order n with variable x, center c." (some people give this polynomial the name "truncated Taylor Series".)

1. Exercises

Consider the function $3x^2 + 3x - 12$, find its Taylor Polynomial centered at $x = 0$ and centered at $x = 1$.

Answer: ($x = 0$) Let $f(x) = 3x^2 + 3x - 12$, then $f(0) = -12, f'(x) = 6x + 3$, therefore $f'(0) = 3$. $f''(x) = 6$, so $f''(0) = 6$. Gathering all these together, we get $f(x) = 3x^2 + 3x - 12 = f(0) + f'(0)x + \frac{f''(0)x^2}{2!}$

$$= -12 + 3x + \frac{6x^2}{2} = the\ function\ we\ started\ with.$$

Remark: Actually, since $3x^2 + 3x - 12$ is a degree 2 polynomial and Taylor polynomials are "better and better" approximation of the original function, it has only up to degree 2 approximation, and the approximating polynomial is itself.

Answer: ($x = 1$). In this case, we can also use brute force computation as above. On the other hand, we can let $y = x - 1$, implying $x = y + 1$. Then we substitute this inside the polynomial $3x^2 + 3x - 12 = 3(y+1)^2 + 3(y+1) - 12 = 3y^2 + 9y + 3 + 3 - 12 = 3(x-1)^2 + 9(x-1) - 6$.

2. Consider the function $\sqrt{1+x}$, find its Taylor Polynomial of this function centered at $x = 0$.
3. Consider the function $\sin x$, find its Taylor Polynomial centered at $x = 0$, and at $x = \pi$.

Some "more" Terminologies

Taylor Series centered at $c = 0$ is called the Maclaurin Series

How to find Taylor Series of a given function

Q: Given a function, infinitely differentiable in an interval centered at

$x = c$, how to find its Taylor Series centered at $x = c$? (we are assuming that the error terms go to zero as $n \to \infty$).

There are at least two methods to do it.

Methods

1) direct method, (long, tedious);

2) smarter methods

To use the second method, we need some tools.

Things to remember:

a) $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots$

b) $\sin x = x - \frac{x^3}{3!} + \cdots$, similar formula hold for $\cos x, e^x, \ln(1+x)$,

Now we give examples to explain Method 2).

Example 1).

Find Taylor Series, centered at $x = 0$ of the function $f(x) = \frac{1}{1-x^2}$

The smarter method makes use of the following facts:

1. $\frac{1}{1-x^2} = \frac{A}{1-x} + \frac{B}{1+x}$, the combining terms on the right-hand side, we get $\frac{1}{1-x^2} =$

$\frac{A}{1-x} + \frac{B}{1+x} = \left(\frac{1}{2}\right)\left\{\frac{1}{1-x} + \frac{1}{1+x}\right\}$

2. Now $\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots$

   $\frac{1}{1-(-x)} = 1 - x + x^2 - \cdots + (-1)^n x^n + \cdots$ therefore we get

   $\left(\frac{1}{2}\right)\left\{\frac{1}{1-x} + \frac{1}{1+x}\right\} = \frac{1}{2}\{1 + x + x^2 + \cdots + x^n + \cdots + (1 - x + x^2 + \cdots +$
   $(-1)^n x^n + \cdots)\}$

   $$= 1 + 1x^2 + 1x^4 + \cdots$$

3. Another method is to let $y = x^2$, then we get $\frac{1}{1-x^2} = \frac{1}{1-y} = 1 + y + y^2 + \cdots$

   $$= 1 + x^2 + x^4 + x^6 + \cdots$$

4. Consider the function $g(x) = \frac{1+x}{1-x}$. Find its Taylor Series centered at $x = 0$.

   Answer: $\frac{1+x}{1-x} = (1 + x)(1 + x + x^2 + \cdots + x^n + \cdots)$

   $= 1 + x + x^2 + \cdots + x^n + \cdots + x(1 + x + x^2 + \cdots + x^n + \cdots)$

5. Consider the function $\frac{\sin x}{1-x}$. Find its Taylor Series centered at $x = 0$.

Some Further Questions

1) What is the domain of $\frac{1}{1-x}$?

2) Is it always possible to rewrite a function (with all derivatives, of course) as a Taylor Series centered at some point?