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DEPARTMENT OF MATHEMATICS

MATH1010J University Mathematics 2019-20
Supplementary Notes on Partial Fractions

Theorem 1. If $Q(x)$ is a polynomial of degree $n \geq 1$, then $Q(x)$ can be factorized as a product of linear and quadratic factors, i.e. $Q(x)$ can be factorized as

$$Q(x) = A(x - \alpha_1)^{p_1} \cdots (x - \alpha_k)^{p_k} (x^2 + b_1x + c_1)^{m_1} \cdots (x^2 + b_r x + c_r)^{m_r},$$

where $p_1 + \cdots + p_k + 2(m_1 + \cdots + m_r) = n$.

If $P(x)$ and $Q(x)$ are polynomials, then the fractions $\frac{P(x)}{Q(x)}$ can be expressed as sum of simpler fractions according to certain rules, the fraction $\frac{P(x)}{Q(x)}$ is said to be resolved in partial fractions. The Rules are listed:

(Rule 0) If $\deg P(x) \geq \deg Q(x)$, by performing long division $P(x) = a(x)Q(x) + b(x)$ where $\deg b(x) < \deg Q(x)$, then

$$\frac{P(x)}{Q(x)} = a(x) + \frac{b(x)}{Q(x)}.$$

Therefore, we only focus on the case that $\deg P(x) < \deg Q(x)$.

(Rule 1) If there is a non-repeating factor $x - a$ of $Q(x)$, then there will be a corresponding partial fraction $\frac{A}{x - a}$.

(Rule 2) If there is a non-repeating factor $x^2 + px + q$ which cannot be further factorized, then there will be a corresponding partial fraction $\frac{Bx + C}{x^2 + px + q}$.

1. Resolve the following expressions into partial fractions.

(a) $\frac{5}{x^2 + x - 6}$ (Hint: $\frac{5}{x^2 + x - 6} \equiv \frac{A}{x + 3} + \frac{B}{x - 2}$)

(b) $\frac{1}{x(x^2 + 1)}$ (Hint: $\frac{1}{x(x^2 + 1)} \equiv \frac{A}{x} + \frac{Bx + C}{x^2 + 1}$)

(c) $\frac{5x^2 - 3x + 4}{(x + 1)(x^2 - 2x + 6)}$

2. Resolve the following expressions into partial fractions.

(a) $\frac{x^2 + 3x}{x^2 + 3x + 2}$

(b) $\frac{x^4 + 2x + 4}{(2x^2 + 3)(x - 2)}$

(c) $\frac{2x^5}{(x^2 - 1)(x^2 - 4)}$

(Rule 3) If $x - a$ is a repeating factor of $Q(x)$ with multiplicity k (i.e. $Q(x) = (x - a)^k Q_1(x)$, where $k > 1$ and $(x - a)$ is not a factor of $Q_1(x)$), then there will be corresponding partial fractions

$$\frac{A_1}{x - a} + \frac{A_2}{(x - a)^2} + \cdots + \frac{A_k}{(x - a)^k}.$$

(Rule 4) If $x^2 + px + q$, which cannot be further factorized, is a repeating factor of $Q(x)$ with multiplicity k , then there will be corresponding partial fractions

$$\frac{B_1x + C_1}{x^2 + px + q} + \frac{B_2x + C_2}{(x^2 + px + q)^2} + \cdots + \frac{B_kx + C_k}{(x^2 + px + q)^k}.$$

3. Resolve the following expressions into partial fractions.

(a) $\frac{x^3 + 1}{(x - 2)^4}$ (Hint: $\frac{x^3 + 1}{(x - 2)^4} \equiv \frac{A}{x - 2} + \frac{B}{(x - 2)^2} + \frac{C}{(x - 2)^3} + \frac{D}{(x - 2)^4}$)

(b) $\frac{2x^2 + 1}{x^2(x^2 + 1)^2}$ (Hint: $\frac{2x^2 + 1}{x^2(x^2 + 1)^2} \equiv \frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{x^2 + 1} + \frac{Ex + F}{(x^2 + 1)^2}$)

4. (a) Resolve $\frac{x^5 + 3x^2 + 1}{(x - 1)(x^2 + 4)}$ into partial fractions.

(b) Hence, evaluate $\int \frac{x^5 + 3x^2 + 1}{(x - 1)(x^2 + 4)} dx$

5. Evaluate $\int \frac{-2x^2 + 6x - 8}{(x - 1)(x^4 - 1)} dx$.

Solution/Hint:

1. (a) $\frac{5}{x^2 + x - 6} = \frac{1}{x - 2} - \frac{1}{x + 3}$

(b) $\frac{1}{x(x^2 + 1)} = \frac{1}{x} - \frac{x}{x^2 + 1}$

(c) $\frac{5x^2 - 3x + 4}{(x + 1)(x^2 - 2x + 6)} = \frac{4}{3(x + 1)} + \frac{11x - 12}{3(x^2 - 2x + 6)}$

2. (a) $\frac{x^2 + 3x}{x^2 + 3x + 2} = 1 - \frac{2}{x + 1} + \frac{2}{x + 2}$

(b) $\frac{x^4 + 2x + 4}{(2x^2 + 3)(x - 2)} = \frac{x}{2} + 1 + \frac{24}{11(x - 2)} - \frac{41x + 38}{22(2x^2 + 3)}$

(c) $\frac{2x^5}{(x^2 - 1)(x^2 - 4)} = 2x - \frac{1}{3(x - 1)} - \frac{1}{3(x + 1)} + \frac{16}{3(x - 2)} + \frac{16}{3(x + 2)}$

3. (a) $\frac{x^3 + 1}{(x - 2)^4} = \frac{9}{(x - 1)^4} + \frac{12}{(x - 2)^3} + \frac{6}{(x - 2)^2} + \frac{1}{x - 2}$

(b) $\frac{2x^2 + 1}{x^2(x^2 + 1)^2} = \frac{1}{x^2} - \frac{1}{x^2 + 1} + \frac{1}{(x^2 + 1)^2}$

4. (a) $\frac{x^5 + 3x^2 + 1}{(x - 1)(x^2 + 4)} = x^2 + x - 3 + \frac{1}{x - 1} + \frac{-x + 15}{x^2 + 4}$

(b) $\int \frac{x^5 + 3x^2 + 1}{(x - 1)(x^2 + 4)} dx = \frac{x^3}{3} + \frac{x^2}{2} - 3x - \frac{1}{2} \ln(x^2 + 4) + \ln|x - 1| + \frac{15}{2} \tan^{-1}\left(\frac{x}{2}\right) + C$

5. $\int \frac{-2x^2 + 6x - 8}{(x - 1)(x^4 - 1)} dx = \frac{1}{x - 1} + 2 \ln|x - 1| - 2 \ln|x + 1| - 3 \tan^{-1} x + C.$