MATH1010H: WEEK 3

ABSTRACT. In this week, we study the *limit of function*. Topics will include *one-side limits*, *limit at infinity*, *algebraic rules for limits*.

1. LIMIT OF FUNCTION

Definition 1.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function. If the value $f(x)$ gets closer and closer to a number L as x gets closer and closer to c from both sides, then L is called the limit of function $f(x)$ at c, and we write*

$$\lim_{x \to c} f(x) = L.$$

- x gets closer and closer to c but $x \neq c$.

Example 1.2. *Let $f(x) = x + 1$. Find $\lim_{x\to 2} f(x)$. How about $\lim_{x\to 3} f(x)$ and $\lim_{x\to c} f(x)$?*

Example 1.3. *Let $f(x) = \frac{x^2-1}{x-1}, x \neq 1$. Find $\lim_{x\to 1} f(x)$.*

Answer: We can write f as the following (piecewise defined function):

$$f(x) = \begin{cases} x+1 & \text{if } x \neq 0 \\ \text{undefined} & \text{if } x = 1 \end{cases}.$$

Thus $f(x)$ tends to 2 as x tends to 1.

Example 1.4. *Let*

$$f(x) = \begin{cases} 1+x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1+x & \text{if } x < 0 \end{cases}.$$

Then $\lim_{x\to 0} f(x)$ does not exists.

One-sided limits:

Definition 1.5. *If $f(x)$ gets closer and closer to a number L (R) as x gets closer and closer to c from the left (right) hand side, then L is called the left (right) hand side limit of $f(x)$ at c. We denote it by*

$$\lim_{x\to c^-} f(x) = L \text{ and } \lim_{x\to c^+} f(x) = R.$$

- the limit of f at c exists if and only if both the left hand side limit and the right hand side limit of f at c exists and equal, that is,

$$\lim_{x\to c} f(x) = k \Longleftrightarrow \lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = k.$$

Example 1.6. *Let*

$$f(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}.$$

Then $\lim_{x\to 0^-} f(x) = -1$ *and* $\lim_{x\to 0^+} = 1$.

- If $f(x) = k$ for all x then $\lim_{x\to c} f(x) = k$. We also write $\lim_{x\to c} k = k$.

Example 1.7. *Let* $f(x) = |x+1| + 5$. *Then*

$$f(x) = \begin{cases} x + 6 & \text{if } x \geqslant -1 \\ -x + 4 & \text{if } x < -1 \end{cases}.$$

Find $\lim_{x\to 1^+} f(x)$ *and* $\lim_{x\to 1^-} f(x)$.

Example 1.8. *Let* $a \in \mathbb{R}$ *and (piece wise defined function)*

$$f(x) = \begin{cases} x^2 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ a\cos x & \text{if } x < 0 \end{cases}.$$

Find the value of a *such that* $\lim_{x\to 0} f(x)$ *exists.*

continuous function
limit at infinity
algebraic rules for limits