Limits of functions

Lecturer: Changhao CHEN

The Chinese University of Hong Kong

18 Feb 2020

Table of Contents

1. Review: sets, functions, limits of sequences and functions

2. New contents: Limits of functions continued

Review: Sets

Definition

A set is a collection of distinct objects.

For instance

$$A = \{a, b, c, d, e, f, \ldots, x, y, z\}.$$

Often used sets:

- $\mathbb{N}$ = natural numbers $= \{1, 2, 3, \ldots\}$
- $\mathbb{Z}$ = integers $= \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
- $\mathbb{Q}$ = rational numbers $= \left\{\frac{m}{n} : m, n \in \mathbb{Z} \text{ and } n \neq 0\right\}$
- $\mathbb{R}$ = set of real numbers
- $\varnothing$ = emptyset $= \{\ \}$
- $[a, b] = \{x \in \mathbb{R} : a \leqslant x \leqslant b\}$

Review: Functions

Definition

A function is a rule that assigns to each element of a set X a single element of a set Y. A function f from X to Y is denoted by

$$f : X \to Y.$$

Review: Functions

Definition

A function is a rule that assigns to each element of a set X a single element of a set Y. A function f from X to Y is denoted by

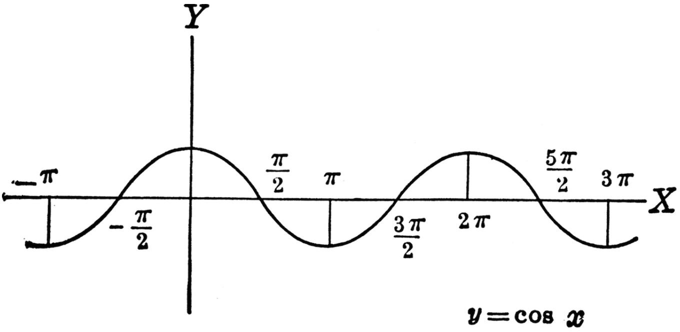
$$f : X \to Y.$$

Graph of a function

Let $f : X \to Y$ be a function. The graph of the function f is defined as

$$G(f) = \{(x, y) : x \in X, y = f(x)\}.$$

Graph of a function



Review: limits of sequences

- A sequence is a list of numbers, that is $a_1, a_2, a_3, \ldots$
- A number L is the limit of the sequence (a_n) if the numbers a_n become closer and closer to L. Denote it as

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L.$$

Review: limits of sequences

- A sequence is a list of numbers, that is $a_1, a_2, a_3, \dots$
- A number L is the limit of the sequence (a_n) if the numbers a_n become closer and closer to L. Denote it as

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L.$$

Note:

$$\lim_{n\to\infty} (-1)^n \qquad \text{does not exists!!}$$

Review: Examples

Example

Find $\lim_{n\to\infty} \frac{n^2+3}{5n^2+7}$.

Review: Examples

Example

Find $\lim_{n\to\infty} \frac{n^2+3}{5n^2+7}$.

Hint: Write

$$\frac{n^2+3}{5n^2+7} = \frac{1+\frac{3}{n^2}}{5+\frac{7}{n^2}}.$$

Review: Examples

Example

Find $\lim_{n\to\infty} \frac{n^2+3}{5n^2+7}$.

Hint: Write

$$\frac{n^2+3}{5n^2+7} = \frac{1+\frac{3}{n^2}}{5+\frac{7}{n^2}}.$$

Example

Prove that $\lim_{n\to\infty}(\sqrt{n+1}-\sqrt{n}) = 0$.

Review: Examples

Example

Find $\lim_{n\to\infty} \frac{n^2+3}{5n^2+7}$.

Hint: Write

$$\frac{n^2+3}{5n^2+7} = \frac{1+\frac{3}{n^2}}{5+\frac{7}{n^2}}.$$

Example

Prove that $\lim_{n\to\infty}(\sqrt{n+1}-\sqrt{n}) = 0$.

Hint: Note

$$\sqrt{n+1}-\sqrt{n} = \frac{1}{\sqrt{n+1}+\sqrt{n}}.$$

Review: limits of functions

Definition

Let $f : \mathbb{R} \to \mathbb{R}$ be a function. If the value $f(x)$ gets closer and closer to a number L as x gets closer and closer to c from both sides, then L is called the limit of function $f(x)$ at c, and we write

$$\lim_{x \to c} f(x) = L.$$

Note:

$$x \text{ tends to } c \Longrightarrow f(x) \text{ tends to } L.$$

Review: limits of functions

Definition

Let $f : \mathbb{R} \to \mathbb{R}$ be a function. If the value $f(x)$ gets closer and closer to a number L as x gets closer and closer to c from both sides, then L is called the limit of function $f(x)$ at c, and we write

$$\lim_{x \to c} f(x) = L.$$

Note:

$$x \text{ tends to } c \Longrightarrow f(x) \text{ tends to } L.$$

Example

Let $f(x) = x + 1$. Find $\lim_{x \to 2} f(x)$ and $\lim_{x \to 3} f(x)$.

Review: limits of functions

Example

Let $f(x) = \frac{x^2-1}{x-1}, x \neq 1$. Find $\lim_{x\to 1} f(x)$.

Review: limits of functions

Example

Let $f(x) = \frac{x^2-1}{x-1}, x \neq 1$. Find $\lim_{x\to 1} f(x)$.

Answer: Write f as the following (piecewise defined function):

$$f(x) = \begin{cases} x+1 & \text{if } x \neq 1 \\ \text{undefined} & \text{if } x = 1 \end{cases}.$$

Thus $f(x)$ tends to 2 as x tends to 1.

Review: limits of functions

Example

Let

$$f(x) = \begin{cases} 1 + x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 + x & \text{if } x < 0 \end{cases}.$$

Then $\lim_{x\to 0} f(x)$ does not exists.

Example

Let

$$f(x) = \begin{cases} 1 + x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 + x & \text{if } x < 0 \end{cases}.$$

Then $\lim_{x \to 0} f(x)$ does not exists.

Answer:

$$\lim_{x \to 0^-} f(x) = -1 \quad \lim_{x \to 0^+} f(x) = 1.$$

Review: limits of functions

- The limit of f at c exists if and only if both the left hand side limit and the right hand side limit of f at c exists and equal, that is,

$$\lim_{x \to c} f(x) = k \iff \lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = k.$$

We arrive to NEW contents from here!

Algebraic properties of limits of functions

Theorem

If both $\lim_{x\to c} f(x)$ *and* $\lim_{x\to c} g(x)$ *exist, then*

$$\lim_{x\to c}(f(x)+g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x);$$

Algebraic properties of limits of functions

Theorem

If both $\lim_{x\to c} f(x)$ *and* $\lim_{x\to c} g(x)$ *exist, then*

$$\lim_{x\to c}(f(x)+g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x);$$

$$\lim_{x\to c}(f(x)-g(x)) = \lim_{x\to c} f(x) - \lim_{x\to c} g(x);$$

Algebraic properties of limits of functions

Theorem

If both $\lim_{x\to c} f(x)$ *and* $\lim_{x\to c} g(x)$ *exist, then*

$$\lim_{x\to c}(f(x)+g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x);$$

$$\lim_{x\to c}(f(x)-g(x)) = \lim_{x\to c} f(x) - \lim_{x\to c} g(x);$$

$$\lim_{x\to c} f(x)g(x) = \lim_{x\to c} f(x) \lim_{x\to c} g(x);$$

Algebraic properties of limits of functions

Theorem

If both $\lim_{x\to c} f(x)$ *and* $\lim_{x\to c} g(x)$ *exist, then*

$$\lim_{x\to c}(f(x)+g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x);$$

$$\lim_{x\to c}(f(x)-g(x)) = \lim_{x\to c} f(x) - \lim_{x\to c} g(x);$$

$$\lim_{x\to c} f(x)g(x) = \lim_{x\to c} f(x) \lim_{x\to c} g(x);$$

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} \quad \textit{if } \lim_{x\to c} g(x) \neq 0.$$

Example I

Example

Find $\lim_{x \to 2} 3x^2 + 5x + 10$.

Example I

Example

Find $\lim_{x\to 2} 3x^2 + 5x + 10$.

Answer:

$$\begin{aligned}\lim_{x\to 2} 3x^2 + 5x + 10 &= \lim_{x\to 2} 3x^2 + \lim_{x\to 2} 5x + \lim_{x\to 2} 10 \\ &= 12 + 10 + 10 = 32.\end{aligned}$$

Example II

Example

Find $\lim_{x \to 1} \frac{3x^2+10}{x^3-10}$.

Example II

Example

Find $\lim_{x\to 1} \frac{3x^2+10}{x^3-10}$.

Answer:

$$\begin{aligned}
\lim_{x\to 1} \frac{3x^2+10}{x^3-10} &= \frac{\lim_{x\to 1}(3x^2+10)}{\lim_{x\to 1}(x^3-10)} \\
&= \frac{\lim_{x\to 1} 3x^2 + \lim_{x\to 1} 10}{\lim_{x\to 1} x^3 - \lim_{x\to 1} 10} \\
&= \frac{3+10}{1-10} = -\frac{13}{9}.
\end{aligned}$$

Example III

Example

Find $\lim_{x\to 1} \frac{x^2-1}{x-1}$.

Note: Since $\lim_{x\to 1}(x-1) = 0$, so we can not use the Theorem.

Example III

Example

Find $\lim_{x\to 1} \frac{x^2-1}{x-1}$.

Note: Since $\lim_{x\to 1}(x-1) = 0$, so we can not use the Theorem.
Answer: For any $x \neq 1$ we have

$$\frac{x^2-1}{x-1} = x+1.$$

Example III

Example

Find $\lim_{x\to 1} \frac{x^2-1}{x-1}$.

Note: Since $\lim_{x\to 1}(x-1) = 0$, so we can not use the Theorem.
Answer: For any $x \neq 1$ we have

$$\frac{x^2-1}{x-1} = x+1.$$

Thus

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1}(x+1) = 2.$$

Relation between limits of sequences and functions

Theorem

$\lim_{x\to c} f(x) = L \iff$ *for any sequence* (a_n) *with* $a_n \neq c$ *and* $\lim_{n\to\infty} a_n = c$ *we have* $\lim_{n\to\infty} f(a_n) = L$.

$$\lim_{x\to c} f(x) = L \iff a_n \to c \text{ implies } f(a_n) \to L$$

Sequential criterion of limits functions

The following methods are very useful for checking whether the limit $\lim_{x \to c} f(x)$ exists or not.

Sequential criterion of limits functions

The following methods are very useful for checking whether the limit $\lim_{x\to c} f(x)$ exists or not.

(1) If $\exists (a_n)$ with $a_n \neq c$ for all $n \in \mathbb{N}$ such that $a_n \to c$, but $\lim_{n\to\infty} f(a_n)$ does not exist, then $\lim_{x\to c} f(x)$ does not exist.

Sequential criterion of limits functions

The following methods are very useful for checking whether the limit $\lim_{x \to c} f(x)$ exists or not.

(1) If $\exists (a_n)$ with $a_n \neq c$ for all $n \in \mathbb{N}$ such that $a_n \to c$, but $\lim_{n \to \infty} f(a_n)$ does not exist, then $\lim_{x \to c} f(x)$ does not exist.

(2) If $\exists (a_n), (b_n)$ with $a_n, b_n \neq c$ for all $n \in \mathbb{N}$ such that $\lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n = c$, but $\lim_{n \to \infty} f(a_n) \neq \lim_{n \to \infty} f(b_n)$, then $\lim_{x \to c} f(x)$ does not exist.

Example I

Example

Consider $\lim_{x\to 0} f(x)$ where $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R}\backslash\mathbb{Q} \end{cases}.$$

Example I

Example

Consider $\lim_{x\to 0} f(x)$ where $f:\mathbb{R}\to\mathbb{R}$ is defined by

$$f(x)=\begin{cases}1 & \text{if } x\in\mathbb{Q}\\ 0 & \text{if } x\in\mathbb{R}\backslash\mathbb{Q}\end{cases}.$$

Method I: Let $a_n=\frac{1}{n}\in\mathbb{Q}$ and $b_n=\frac{\sqrt{2}}{n}\in\mathbb{R}\backslash\mathbb{Q}$. Then

$$\lim_{n\to\infty} a_n=\lim_{n\to\infty} b_n=0,$$

Example I

Example

Consider $\lim_{x\to 0} f(x)$ where $f:\mathbb{R}\to\mathbb{R}$ is defined by

$$f(x)=\begin{cases}1 & \text{if } x\in\mathbb{Q}\\ 0 & \text{if } x\in\mathbb{R}\backslash\mathbb{Q}\end{cases}.$$

Method I: Let $a_n=\frac{1}{n}\in\mathbb{Q}$ and $b_n=\frac{\sqrt{2}}{n}\in\mathbb{R}\backslash\mathbb{Q}$. Then

$$\lim_{n\to\infty} a_n=\lim_{n\to\infty} b_n=0,$$

but

$$\lim_{n\to\infty} f(a_n)=\lim_{n\to\infty} f\left(\frac{1}{n}\right)=1,\quad \lim_{n\to\infty} f(b_n)=\lim_{n\to\infty} f\left(\frac{\sqrt{2}}{n}\right)=0.$$

Example I continued

Example

Consider $\lim_{x \to 0} f(x)$ where $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \backslash \mathbb{Q} \end{cases}.$$

Example I continued

Example

Consider $\lim_{x\to 0} f(x)$ where $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R}\backslash\mathbb{Q} \end{cases}.$$

Method 2: We define a sequence in the following way. For $k \in \mathbb{N}$ let $a_{2k} = \frac{1}{k}$ and $a_{2k+1} = \frac{\sqrt{2}}{k}$. Thus we have $\lim_{n\to\infty} a_n = 0$.

Example I continued

Example

Consider $\lim_{x\to 0} f(x)$ where $f:\mathbb{R}\to\mathbb{R}$ is defined by

$$f(x)=\begin{cases}1 & \text{if } x\in\mathbb{Q}\\ 0 & \text{if } x\in\mathbb{R}\backslash\mathbb{Q}\end{cases}.$$

Method 2: We define a sequence in the following way. For $k\in\mathbb{N}$ let $a_{2k}=\frac{1}{k}$ and $a_{2k+1}=\frac{\sqrt{2}}{k}$. Thus we have $\lim_{n\to\infty} a_n = 0$. But $\lim_{n\to\infty} f(a_n)$ does not exists, because of

$$f(a_{2k})=f\left(\frac{1}{k}\right)=1,\quad f(a_{2k+1})=f\left(\frac{\sqrt{2}}{k}\right)=0.$$

Example II

Example

Let $f(x) = \sin \frac{1}{x}$ for $x \neq 0$. Show that $\lim_{x \to 0} f(x)$ does not exist.

Example II

Example

Let $f(x) = \sin\frac{1}{x}$ for $x \neq 0$. Show that $\lim_{x\to 0} f(x)$ does not exist.

Answer: Let $a_n = \frac{1}{2\pi n}$ and $b_n = \frac{1}{2n\pi + \frac{\pi}{2}}$. Then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0,$$

Example II

Example

Let $f(x) = \sin\frac{1}{x}$ for $x \neq 0$. Show that $\lim_{x\to 0} f(x)$ does not exist.

Answer: Let $a_n = \frac{1}{2\pi n}$ and $b_n = \frac{1}{2n\pi + \frac{\pi}{2}}$. Then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0,$$

but

$$\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} \sin(2\pi n) = 0, \quad \lim_{n\to\infty} f(b_n) = \lim_{n\to\infty} \sin(2n\pi + \frac{\pi}{2}) = 1.$$

Example II

Example

Let $f(x) = \sin \frac{1}{x}$ for $x \neq 0$. Show that $\lim_{x \to 0} f(x)$ does not exist.

Example II

Example

Let $f(x) = \sin\frac{1}{x}$ for $x \neq 0$. Show that $\lim_{x\to 0} f(x)$ does not exist.