Differential I: Definition and basic rules

Lecturer: Changhao CHEN

Some figures comes from Dr. CHAN Kai Leung, thanks a lot!!

The Chinese University of Hong Kong

21 Feb 2020

A tip first

Four key words in our course:

- Limits
- Continuous function
- Differentiation — around 5 weeks
- Integration— around 5 weeks

Motivation

Motivation

$$\text{Average speed} = \frac{\text{distance}}{\text{time}} = \frac{100}{\text{Your time}}.$$

Motivation

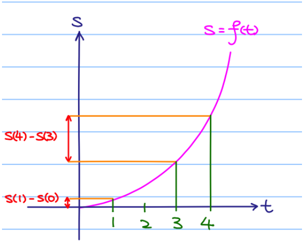
$$\text{Average speed} = \frac{\text{distance}}{\text{time}} = \frac{100}{\text{Your time}}.$$

The average speed between t_1 and t_2?

$$\text{Average speed on } [t_1, t_2] = \frac{S(t_2) - S(t_1)}{t_2 - t_1}.$$

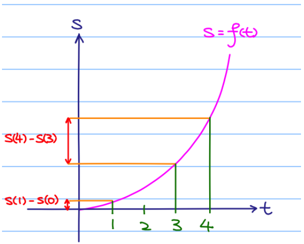
Motivation

Let $S(t) = f(t)$ be a function (position).



Motivation

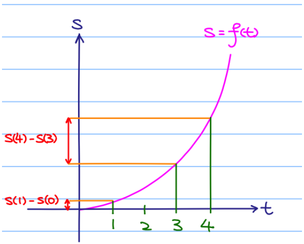
Let $S(t) = f(t)$ be a function (position).



The distance travel from $t = 0$ to $t = 1$ is less than the distance travel from $t = 3$ to $t = 4$. That is $S(2) - S(1) < S(4) - S(3)$. The speed is different on these two intervals.

Motivation

Let $S(t) = f(t)$ be a function (position).



The distance travel from $t = 0$ to $t = 1$ is less than the distance travel from $t = 3$ to $t = 4$. That is $S(2) - S(1) < S(4) - S(3)$. The speed is different on these two intervals.

The average speed at interval $[t_0, t_0 + \Delta t]$ is $\frac{S(t_0+\Delta t)-S(\Delta t)}{\Delta t}$.

Motivation: Instantaneous speed at time t_0

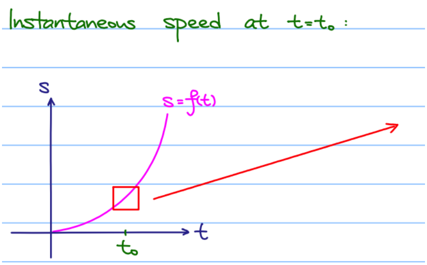
When Δt becomes smaller and smaller, we obtain the instantaneous speed at time t_0, i.e.,

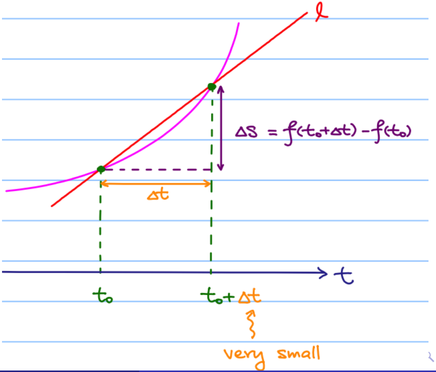
$$\lim_{\Delta t \to 0} \frac{S(t_0 + \Delta t) - S(t_0)}{\Delta t}$$

Motivation: Instantaneous speed at time t_0

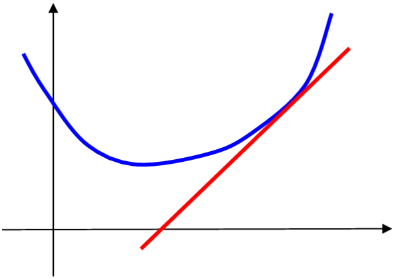
When Δt becomes smaller and smaller, we obtain the instantaneous speed at time t_0, i.e.,

$$\lim_{\Delta t \to 0} \frac{S(t_0 + \Delta t) - S(t_0)}{\Delta t}$$

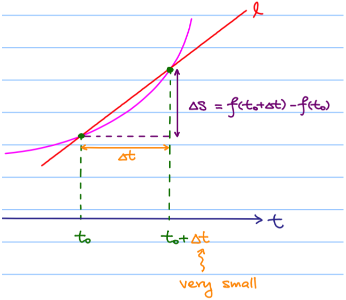


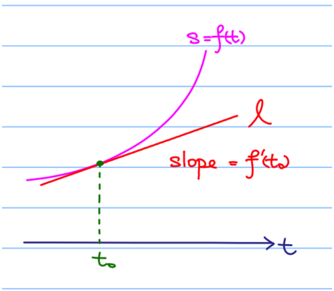


Definition: tangent line



Motivation: Tangent line at point $(t_0, S(t_0))$

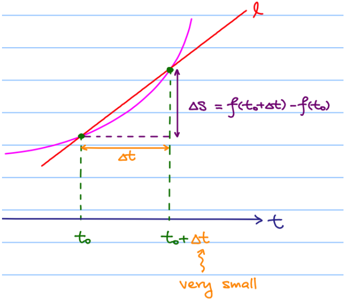


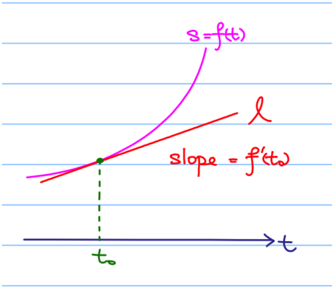


When Δt becomes smaller and smaller, we obtain the tangent line at point $(t_0, S(t_0))$ with slope

$$\lim_{\Delta t \to 0} \frac{S(t + \Delta t) - S(t)}{\Delta t}.$$

Motivation: Tangent line at point $(t_0, S(t_0))$





When Δt becomes smaller and smaller, we obtain the tangent line at point $(t_0, S(t_0))$ with slope

$$\lim_{\Delta t \to 0} \frac{S(t + \Delta t) - S(t)}{\Delta t}.$$

OK, it is the time for the definition now!!

Differential function

Definition

Let $x_0 \in \mathcal{D} \subseteq \mathbb{R}$ and let $f : \mathcal{D} \subseteq \mathbb{R}$ be a function. We say that function $f(x)$ is differential at the point x_0 if the limit

$$\lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

exists (called the first principle). It is called the derivative of $f(x)$ at $x = x_0$ and it is denoted by $f'(x_0)$.

Differential function

Definition

Let $x_0 \in \mathcal{D} \subseteq \mathbb{R}$ and let $f : \mathcal{D} \subseteq \mathbb{R}$ be a function. We say that function $f(x)$ is differential at the point x_0 if the limit

$$\lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

exists (called the first principle). It is called the derivative of $f(x)$ at $x = x_0$ and it is denoted by $f'(x_0)$. We say that $f(x)$ is a differential function if f is differentiable at every point in $\mathcal{D}$.

Differential function

Definition

Let $x_0 \in \mathcal{D} \subseteq \mathbb{R}$ and let $f : \mathcal{D} \subseteq \mathbb{R}$ be a function. We say that function $f(x)$ is differential at the point x_0 if the limit

$$\lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

exists (called the first principle). It is called the derivative of $f(x)$ at $x = x_0$ and it is denoted by $f'(x_0)$. We say that $f(x)$ is a differential function if f is differentiable at every point in $\mathcal{D}$.

Note: The set $\mathcal{D}$ will be one of the following sets:

$$\mathbb{R}, (a, b), (t, +\infty), (-\infty, s)$$

Differential at some point x_0

By the definition, $f(x)$ is differentiable at the point x_0 if the limit

$$\lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

exits. Let $\Delta x = x - x_0$, then $\Delta x \to 0$ if and only if $x \to x_0$.

Differential at some point x_0

By the definition, $f(x)$ is differentiable at the point x_0 if the limit

$$\lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

exits. Let $\Delta x = x - x_0$, then $\Delta x \to 0$ if and only if $x \to x_0$.

Therefore, we have another form: $f(x)$ is differentiable at $x = x_0$ if the following limit exists,

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

Notations about differential

Note: Give a differential function f, we can consider the derivative $f'(x)$ as a function.

Notations about differential

Note: Give a differential function f, we can consider the derivative $f'(x)$ as a function.

- For $y = f(x)$ the derivative function of f is often denoted as

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx}.$$

Notations about differential

Note: Give a differential function f, we can consider the derivative $f'(x)$ as a function.

- For $y = f(x)$ the derivative function of f is often denoted as
$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx}.$$
- Given a fixed value x_0 the derivative function of f at $x = x_0$ is often denoted as
$$f'(x_0) = y'\Big|_{x=x_0} = \frac{dy}{dx}\Big|_{x=x_0} = \frac{df}{dx}\Big|_{x=x_0}.$$

The study plan for differential

1. Definitions, examples, work out the derivative function for the elementary functions.

The study plan for differential

1. Definitions, examples, work out the derivative function for the elementary functions. For instance, what are the derivative functions of

$$x^n, \sin x, \cos x, \tan x, a^x, \log_a x, \ln x, \frac{x+3}{x^2+5}, \ldots$$

The study plan for differential

1. Definitions, examples, work out the derivative function for the elementary functions. For instance, what are the derivative functions of

$$x^n, \sin x, \cos x, \tan x, a^x, \log_a x, \ln x, \frac{x+3}{x^2+5}, \ldots$$

2. Properties of differential functions: if f is a differential function then f is a continuous function. But the reverse is not true

The study plan for differential

1. Definitions, examples, work out the derivative function for the elementary functions. For instance, what are the derivative functions of

$$x^n, \sin x, \cos x, \tan x, a^x, \log_a x, \ln x, \frac{x+3}{x^2+5}, \ldots$$

2. Properties of differential functions: if f is a differential function then f is a continuous function. But the reverse is not true
3. Applications of differential—Friday and the next week

The study plan for differential

1. Definitions, examples, work out the derivative function for the elementary functions. For instance, what are the derivative functions of

$$x^n, \sin x, \cos x, \tan x, a^x, \log_a x, \ln x, \frac{x+3}{x^2+5}, \ldots$$

2. Properties of differential functions: if f is a differential function then f is a continuous function. But the reverse is not true
3. Applications of differential—Friday and the next week
4. Taylor theorem and its applications

Let $f(x) = k$. Find $f'(x)$.

Let $f(x) = k$. Find $f'(x)$.

Let $x \in \mathbb{R}$. Since for any Δx, $f(x + \Delta x) - f(x) = 0$,

Let $f(x) = k$. Find $f'(x)$.

Let $x \in \mathbb{R}$. Since for any Δx, $f(x + \Delta x) - f(x) = 0$, we obtain

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = 0.$$

Let $f(x) = k$. Find $f'(x)$.

Let $x \in \mathbb{R}$. Since for any Δx, $f(x + \Delta x) - f(x) = 0$, we obtain

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = 0.$$

Thus $f'(x) = 0$.

Let $f(x) = k$. Find $f'(x)$.

Let $x \in \mathbb{R}$. Since for any Δx, $f(x + \Delta x) - f(x) = 0$, we obtain

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = 0.$$

Thus $f'(x) = 0$.
We also write it as

$$f'(x) = \frac{dy}{dx} = \frac{df}{dx} = 0$$

Let $f(x) = ax, a \neq 0$. Find $f'(x)$.

Let $f(x) = ax, a \neq 0$. Find $f'(x)$.

For $x \in \mathbb{R}$ and any Δx we have

$$f(x + \Delta x) - f(x) = a(x + \Delta x) - a\Delta x = a\Delta x.$$

Let $f(x) = ax, a \neq 0$. Find $f'(x)$.

For $x \in \mathbb{R}$ and any Δx we have

$$f(x + \Delta x) - f(x) = a(x + \Delta x) - a\Delta x = a\Delta x.$$

This gives

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{a\Delta x}{\Delta x} = a.$$

Let $f(x) = ax, a \neq 0$. Find $f'(x)$.

For $x \in \mathbb{R}$ and any Δx we have

$$f(x + \Delta x) - f(x) = a(x + \Delta x) - a\Delta x = a\Delta x.$$

This gives

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{a\Delta x}{\Delta x} = a.$$

Thus $f'(x) = a$.

Let $f(x) = ax, a \neq 0$. Find $f'(x)$.

For $x \in \mathbb{R}$ and any Δx we have

$$f(x + \Delta x) - f(x) = a(x + \Delta x) - a\Delta x = a\Delta x.$$

This gives

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{a\Delta x}{\Delta x} = a.$$

Thus $f'(x) = a$.

$$f'(x) = \frac{dy}{dx} = \frac{df}{dx} = a.$$

Let $f(x) = x^2$. Find $f'(x)$.

Let $f(x) = x^2$. Find $f'(x)$.

For $x \in \mathbb{R}$ and any Δx we have

$$f(x + \Delta x) - f(x) = (x + \Delta x)^2 - x^2 = 2x\Delta x + (\Delta x)^2.$$

Let $f(x) = x^2$. Find $f'(x)$.

For $x \in \mathbb{R}$ and any Δx we have

$$f(x + \Delta x) - f(x) = (x + \Delta x)^2 - x^2 = 2x\Delta x + (\Delta x)^2.$$

This gives

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x}$$

Let $f(x) = x^2$. Find $f'(x)$.

For $x \in \mathbb{R}$ and any Δx we have

$$f(x+\Delta x) - f(x) = (x+\Delta x)^2 - x^2 = 2x\Delta x + (\Delta x)^2.$$

This gives

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (2x + \Delta x) = 2x.$$

Let $f(x) = x^2$. Find $f'(x)$.

For $x \in \mathbb{R}$ and any Δx we have

$$f(x+\Delta x) - f(x) = (x+\Delta x)^2 - x^2 = 2x\Delta x + (\Delta x)^2.$$

This gives

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (2x + \Delta x) = 2x.$$

Thus $f'(x) = 2x$.

Let $f(x) = x^2$. Find $f'(x)$.

For $x \in \mathbb{R}$ and any Δx we have

$$f(x+\Delta x) - f(x) = (x+\Delta x)^2 - x^2 = 2x\Delta x + (\Delta x)^2.$$

This gives

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (2x + \Delta x) = 2x.$$

Thus $f'(x) = 2x$. Also denote it as $\frac{dy}{dx} = \frac{df}{dx} = 2x$.

Let $f(x) = x^2$. Find $f'(x)$.

For $x \in \mathbb{R}$ and any Δx we have

$$f(x + \Delta x) - f(x) = (x + \Delta x)^2 - x^2 = 2x\Delta x + (\Delta x)^2.$$

This gives

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (2x + \Delta x) = 2x.$$

Thus $f'(x) = 2x$. Also denote it as $\frac{dy}{dx} = \frac{df}{dx} = 2x$. Moreover,

$$f'(3) = \left.\frac{dx^2}{dx}\right|_{x=3} = \left.\frac{df}{dx}\right|_{x=3} = 6.$$

Let $f(x) = x^3$. Find $f'(x)$.

Let $f(x) = x^3$. Find $f'(x)$.

For $x \in \mathbb{R}$ and any Δx we have

$$f(x + \Delta x) - f(x) = (x + \Delta x)^3 - x^3 = 3x^2 \Delta x + 3x(\Delta x)^2 + (\Delta x)^3.$$

Let $f(x) = x^3$. Find $f'(x)$.

For $x \in \mathbb{R}$ and any Δx we have

$$f(x+\Delta x) - f(x) = (x+\Delta x)^3 - x^3 = 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3.$$

This gives

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3}{\Delta x}$$

Let $f(x) = x^3$. Find $f'(x)$.

For $x \in \mathbb{R}$ and any Δx we have

$$f(x+\Delta x) - f(x) = (x+\Delta x)^3 - x^3 = 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3.$$

This gives

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (3x^2 + 3x\Delta x + (\Delta x)^2) = 3x^2.$$

Let $f(x) = x^3$. Find $f'(x)$.

For $x \in \mathbb{R}$ and any Δx we have

$$f(x+\Delta x) - f(x) = (x+\Delta x)^3 - x^3 = 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3.$$

This gives

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (3x^2 + 3x\Delta x + (\Delta x)^2) = 3x^2.$$

Thus $f'(x) = 3x^2$.

Let $f(x) = x^3$. Find $f'(x)$.

For $x \in \mathbb{R}$ and any Δx we have

$$f(x+\Delta x) - f(x) = (x+\Delta x)^3 - x^3 = 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3.$$

This gives

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (3x^2 + 3x\Delta x + (\Delta x)^2) = 3x^2.$$

Thus $f'(x) = 3x^2$. Also denote it as $\frac{dy}{dx} = \frac{df}{dx} = 3x^2$.

Let $f(x) = x^3$. Find $f'(x)$.

For $x \in \mathbb{R}$ and any Δx we have

$$f(x + \Delta x) - f(x) = (x + \Delta x)^3 - x^3 = 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3.$$

This gives

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (3x^2 + 3x\Delta x + (\Delta x)^2) = 3x^2.$$

Thus $f'(x) = 3x^2$. Also denote it as $\frac{dy}{dx} = \frac{df}{dx} = 3x^2$. Moreover,

$$f'(3) = \left.\frac{dx^3}{dx}\right|_{x=3} = \left.\frac{df}{dx}\right|_{x=3} = 27.$$

Derivative of monomials

Now we proved

- $(k)' = 0$

Derivative of monomials

Now we proved

- $(k)' = 0$
- $(x)' = 1$

Derivative of monomials

Now we proved

- $(k)' = 0$
- $(x)' = 1$
- $(x^2)' = 2x$

Derivative of monomials

Now we proved

- $(k)' = 0$
- $(x)' = 1$
- $(x^2)' = 2x$
- $(x^3)' = 3x^2$

Derivative of monomials

Now we proved

- $(k)' = 0$
- $(x)' = 1$
- $(x^2)' = 2x$
- $(x^3)' = 3x^2$
- Can you guess $(x^4)' = ?$ and $(x^n)' = ?$.

Derivative of monomials

Now we proved

- $(k)' = 0$
- $(x)' = 1$
- $(x^2)' = 2x$
- $(x^3)' = 3x^2$
- Can you guess $(x^4)' =?$ and $(x^n)' =?$.

Theorem

Let $f(x) = x^r$ for some constant $r \in \mathbb{R}$, then $f'(x) = rx^{r-1}$ whenever it is defined.

Problems

- What's the derivative functions of x^4 and x^5?

Problems

- What's the derivative functions of x^4 and x^5?
- What's the derivative function of $x^4 + x^5$?

Problems

- What's the derivative functions of x^4 and x^5?
- What's the derivative function of $x^4 + x^5$?
- Find

$$\frac{d}{dx}(x^4 - x^5 + 7)?$$

Problems

- What's the derivative functions of x^4 and x^5?
- What's the derivative function of $x^4 + x^5$?
- Find
$$\frac{d}{dx}(x^4 - x^5 + 7)?$$
- Find
$$\frac{d}{dx}\frac{2x}{x^2+1}?$$

Rules of computation

Theorem

If $f(x)$ and $g(x)$ are differential functions, then

Rules of computation

Theorem

If $f(x)$ and $g(x)$ are differential functions, then

$$(f+g)'(x) = f'(x) + g'(x);$$

Rules of computation

Theorem

If $f(x)$ and $g(x)$ are differential functions, then

$$(f+g)'(x) = f'(x) + g'(x);$$

$$(f-g)'(x) = f'(x) - g'(x);$$

Rules of computation

Theorem

If $f(x)$ and $g(x)$ are differential functions, then

$$(f+g)'(x) = f'(x) + g'(x);$$

$$(f-g)'(x) = f'(x) - g'(x);$$

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x); \quad \textit{Product rule}$$

Rules of computation

Theorem

If $f(x)$ and $g(x)$ are differential functions, then

$$(f+g)'(x) = f'(x) + g'(x);$$

$$(f-g)'(x) = f'(x) - g'(x);$$

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x); \quad \textit{Product rule}$$

$$\left(\frac{f}{g}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}, \quad \textit{if} \quad g(x) \neq 0. \quad \textit{Quotient rule}$$

Proof of $(f+g)' = f' + g'$

Let $F(x) = f(x) + g(x)$ then using definition of differential.

Proof of $(f+g)' = f' + g'$

Let $F(x) = f(x) + g(x)$ then using definition of differential. We have

$$F(x+\Delta x) - F(x) = f(x+\Delta x) - f(x) + g(x+\Delta x) - g(x),$$

Proof of $(f+g)' = f' + g'$

Let $F(x) = f(x) + g(x)$ then using definition of differential. We have

$$F(x+\Delta x) - F(x) = f(x+\Delta x) - f(x) + g(x+\Delta x) - g(x),$$

and hence

$$\begin{aligned}
&\lim_{\Delta x \to 0} \frac{F(x+\Delta x) - F(x)}{\Delta x} \\
= &\lim_{\Delta x \to 0} \left(\frac{f(x+\Delta x) - f(x)}{\Delta x} + \frac{g(x+\Delta x) - g(x)}{\Delta x} \right)
\end{aligned}$$

Proof of $(f+g)' = f' + g'$

Let $F(x) = f(x) + g(x)$ then using definition of differential. We have

$$F(x+\Delta x) - F(x) = f(x+\Delta x) - f(x) + g(x+\Delta x) - g(x),$$

and hence

$$\begin{aligned}
&\lim_{\Delta x \to 0} \frac{F(x+\Delta x) - F(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left(\frac{f(x+\Delta x) - f(x)}{\Delta x} + \frac{g(x+\Delta x) - g(x)}{\Delta x} \right) \\
&= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} + \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x}
\end{aligned}$$

Proof of $(f+g)' = f' + g'$

Let $F(x) = f(x) + g(x)$ then using definition of differential. We have

$$F(x+\Delta x) - F(x) = f(x+\Delta x) - f(x) + g(x+\Delta x) - g(x),$$

and hence

$$\begin{aligned}
&\lim_{\Delta x\to 0} \frac{F(x+\Delta x) - F(x)}{\Delta x} \\
&= \lim_{\Delta x\to 0} \left(\frac{f(x+\Delta x) - f(x)}{\Delta x} + \frac{g(x+\Delta x) - g(x)}{\Delta x} \right) \\
&= \lim_{\Delta x\to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} + \lim_{\Delta x\to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x} \\
&= f'(x) + g'(x).
\end{aligned}$$

Example

Find $\frac{d}{dx}(x^4 + x^5)$.

Example

Find $\frac{d}{dx}(x^4 + x^5)$.

Hint: $(f+g)' = f' + g'$

Example

Find $\frac{d}{dx}(x^4 + x^5)$.

Hint: $(f+g)' = f' + g'$

$$\frac{d}{dx}(x^4 + x^5) = \frac{d}{dx}x^4 + \frac{d}{dx}x^5$$

Example

Find $\frac{d}{dx}(x^4 + x^5)$.

Hint: $(f + g)' = f' + g'$

$$\frac{d}{dx}(x^4 + x^5) = \frac{d}{dx}x^4 + \frac{d}{dx}x^5$$

$$= 4x^3 + 5x^4.$$

Example

Find $\frac{d}{dx}(a_0 + a_1x + a_2x^2 + \dots a_nx^n)$.

Example

Find $\frac{d}{dx}(a_0 + a_1 x + a_2 x^2 + \ldots a_n x^n)$.

Hint: $f + g + h = f + (g + h)$. Thus

$$(f + g + h)' = (f + (g + h))' = f' + (f + g)' = f' + g' + h'.$$

Example

Find $\frac{d}{dx}(a_0 + a_1 x + a_2 x^2 + \ldots a_n x^n)$.

Hint: $f + g + h = f + (g + h)$. Thus

$$(f + g + h)' = (f + (g + h))' = f' + (f + g)' = f' + g' + h'.$$

Then

$$\left(\sum_{k=0}^{n} a_k x^k\right)' = \sum_{k=0}^{n} (a_k x^k)'$$

Example

Find $\frac{d}{dx}(a_0 + a_1 x + a_2 x^2 + \dots a_n x^n)$.

Hint: $f + g + h = f + (g + h)$. Thus

$$(f + g + h)' = (f + (g + h))' = f' + (f + g)' = f' + g' + h'.$$

Then

$$\left(\sum_{k=0}^{n} a_k x^k\right)' = \sum_{k=0}^{n} (a_k x^k)'$$

$$= a_1 + 2a_2 x + 3a_3 x^2 + \dots n a_n x^{n-1}.$$

Example

Find $\frac{d}{dx}(x+1)(x+2)$.

Example

Find $\frac{d}{dx}(x+1)(x+2)$.

Hint: $(fg)' = f'g + fg'$.

Example

Find $\frac{d}{dx}(x+1)(x+2)$.

Hint: $(fg)' = f'g + fg'$.

$$\frac{d}{dx}(x+1)(x+2) = (x+1)'(x+2) + (x+1)(x+2)'$$

Example

Find $\frac{d}{dx}(x+1)(x+2)$.

Hint: $(fg)' = f'g + fg'$.

$$\frac{d}{dx}(x+1)(x+2) = (x+1)'(x+2) + (x+1)(x+2)'$$

$$= x+2+x+1 = 2x+3.$$

Example

Find $\frac{d}{dx}(x+1)(x+2)$.

Hint: $(fg)' = f'g + fg'$.

$$\frac{d}{dx}(x+1)(x+2) = (x+1)'(x+2) + (x+1)(x+2)'$$

$$= x+2+x+1 = 2x+3.$$

Also note that

$$\frac{d}{dx}(x+1)(x+2) = \frac{d}{dx}(x^2+3x+2)$$

Example

Find $\frac{d}{dx}(x+1)(x+2)$.

Hint: $(fg)' = f'g + fg'$.

$$\frac{d}{dx}(x+1)(x+2) = (x+1)'(x+2) + (x+1)(x+2)'$$

$$= x + 2 + x + 1 = 2x + 3.$$

Also note that

$$\frac{d}{dx}(x+1)(x+2) = \frac{d}{dx}(x^2+3x+2)$$

$$= \frac{d}{dx}x^2 + \frac{d}{dx}3x + \frac{d}{dx}2$$

Example

Find $\frac{d}{dx}(x+1)(x+2)$.

Hint: $(fg)' = f'g + fg'$.

$$\frac{d}{dx}(x+1)(x+2) = (x+1)'(x+2) + (x+1)(x+2)'$$

$$= x+2+x+1 = 2x+3.$$

Also note that

$$\frac{d}{dx}(x+1)(x+2) = \frac{d}{dx}(x^2+3x+2)$$

$$= \frac{d}{dx}x^2 + \frac{d}{dx}3x + \frac{d}{dx}2$$

$$= 2x+3+0$$

Example

Find $\frac{d}{dx}(x+1)(x+2)$.

Hint: $(fg)' = f'g + fg'$.

$$\frac{d}{dx}(x+1)(x+2) = (x+1)'(x+2) + (x+1)(x+2)'$$

$$= x + 2 + x + 1 = 2x + 3.$$

Also note that

$$\frac{d}{dx}(x+1)(x+2) = \frac{d}{dx}(x^2+3x+2)$$

$$= \frac{d}{dx}x^2 + \frac{d}{dx}3x + \frac{d}{dx}2$$

$$= 2x + 3 + 0$$

$$= 2x + 3$$

Example

Find $\frac{d}{dx}\left(\frac{2x}{x^2+1}\right)$.

Example

Find $\frac{d}{dx}\left(\frac{2x}{x^2+1}\right)$.

Hint: $\frac{f}{g} = \frac{f'g-fg'}{g^2}$.

Example

Find $\frac{d}{dx}\left(\frac{2x}{x^2+1}\right)$.

Hint: $\frac{f}{g} = \frac{f'g - fg'}{g^2}$.

$$\left(\frac{2x}{x^2+1}\right)' = \frac{(2x)'(x^2+1) - 2x(x^2+1)'}{(x^2+1)^2}$$

Example

Find $\frac{d}{dx}\left(\frac{2x}{x^2+1}\right)$.

Hint: $\frac{f}{g} = \frac{f'g-fg'}{g^2}$.

$$\left(\frac{2x}{x^2+1}\right)' = \frac{(2x)'(x^2+1) - 2x(x^2+1)'}{(x^2+1)^2}$$

$$= \frac{2(x^2+1) - 2x(2x)}{(x^2+1)^2}$$

Example

Find $\frac{d}{dx}\left(\frac{2x}{x^2+1}\right)$.

Hint: $\frac{f}{g} = \frac{f'g-fg'}{g^2}$.

$$\left(\frac{2x}{x^2+1}\right)' = \frac{(2x)'(x^2+1) - 2x(x^2+1)'}{(x^2+1)^2}$$

$$= \frac{2(x^2+1) - 2x(2x)}{(x^2+1)^2}$$

$$= \frac{1-2x^2}{(x^2+1)^2}.$$

Theorem

Let $f(x) = x^r$ for some constant $r \in \mathbb{R}$, then $f'(x) = rx^{r-1}$ whenever it is defined.

Proof. We prove $f(x) = x^{\frac{1}{2}}$ only, i.e., $(x^{1/2})' = \frac{1}{2}x^{-1/2}$.

Theorem

Let $f(x) = x^r$ for some constant $r \in \mathbb{R}$, then $f'(x) = rx^{r-1}$ whenever it is defined.

Proof. We prove $f(x) = x^{\frac{1}{2}}$ only, i.e., $(x^{1/2})' = \frac{1}{2}x^{-1/2}$. For x and Δx we have

$$\sqrt{x + \Delta x} - \sqrt{x} = \frac{(\sqrt{x + \Delta x} - \sqrt{x})(\sqrt{x + \Delta x} + \sqrt{x})}{\sqrt{x + \Delta x} + \sqrt{x}}$$

Theorem

Let $f(x) = x^r$ for some constant $r \in \mathbb{R}$, then $f'(x) = rx^{r-1}$ whenever it is defined.

Proof. We prove $f(x) = x^{\frac{1}{2}}$ only, i.e., $(x^{1/2})' = \frac{1}{2}x^{-1/2}$. For x and Δx we have

$$\sqrt{x+\Delta x} - \sqrt{x} = \frac{(\sqrt{x+\Delta x} - \sqrt{x})(\sqrt{x+\Delta x} + \sqrt{x})}{\sqrt{x+\Delta x} + \sqrt{x}}$$

$$= \frac{\Delta x}{\sqrt{x+\Delta x} + \sqrt{x}}.$$

Theorem

Let $f(x) = x^r$ for some constant $r \in \mathbb{R}$, then $f'(x) = rx^{r-1}$ whenever it is defined.

Proof. We prove $f(x) = x^{\frac{1}{2}}$ only, i.e., $(x^{1/2})' = \frac{1}{2}x^{-1/2}$. For x and Δx we have

$$\sqrt{x+\Delta x} - \sqrt{x} = \frac{(\sqrt{x+\Delta x} - \sqrt{x})(\sqrt{x+\Delta x} + \sqrt{x})}{\sqrt{x+\Delta x} + \sqrt{x}}$$

$$= \frac{\Delta x}{\sqrt{x+\Delta x} + \sqrt{x}}.$$

Thus

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{1}{\sqrt{x+\Delta x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}.$$

Proof of $(fg)' = f'g + g'f$ I

Proof of $(fg)' = f'g + g'f$ I

We use the following

$$f(x+h)g(x+h) - f(x)g(x)$$

$$= f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)$$

Proof of $(fg)' = f'g + g'f$ I

We use the following

$$f(x+h)g(x+h) - f(x)g(x)$$

$$= f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)$$

$$= (f(x+h) - f(x))g(x+h) + f(x)(g(x+h) - g(x)).$$

Proof of $(fg)' = f'g + g'f$ I

We use the following

$$f(x+h)g(x+h) - f(x)g(x)$$

$$= f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)$$

$$= (f(x+h) - f(x))g(x+h) + f(x)(g(x+h) - g(x)).$$

Then using the definition of differential, limit rules, differential function is continuous, we obtain

Proof of $(fg)' = f'g + g'f$ I

We use the following

$$f(x+h)g(x+h) - f(x)g(x)$$

$$= f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)$$

$$= (f(x+h) - f(x))g(x+h) + f(x)(g(x+h) - g(x)).$$

Then using the definition of differential, limit rules, differential function is continuous, we obtain

$$\lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

Proof of $(fg)' = f'g + g'f$ I

We use the following

$$f(x+h)g(x+h) - f(x)g(x)$$

$$= f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)$$

$$= (f(x+h) - f(x))g(x+h) + f(x)(g(x+h) - g(x)).$$

Then using the definition of differential, limit rules, differential function is continuous, we obtain

$$\lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \left(\frac{(f(x+h) - f(x))g(x+h)}{h} + \frac{f(x)(g(x+h) - g(x))}{h} \right)$$

Proof of $(fg)' = f'g + g'f$ II

$$= \lim_{h\to 0} \frac{(f(x+h) - f(x))g(x+h)}{h} + \lim_{h\to 0} \frac{f(x)(g(x+h) - g(x))}{h}$$

Proof of $(fg)' = f'g + g'f$ II

$$= \lim_{h\to 0} \frac{(f(x+h) - f(x))g(x+h)}{h} + \lim_{h\to 0} \frac{f(x)(g(x+h) - g(x))}{h}$$

$$= \lim_{h\to 0} \frac{(f(x+h) - f(x))}{h} \lim_{h\to 0} g(x+h)$$

Proof of $(fg)' = f'g + g'f$ II

$$= \lim_{h\to 0}\frac{(f(x+h)-f(x))g(x+h)}{h} + \lim_{h\to 0}\frac{f(x)(g(x+h)-g(x))}{h}$$

$$= \lim_{h\to 0}\frac{(f(x+h)-f(x))}{h}\lim_{h\to 0} g(x+h)$$

$$+ \lim_{h\to 0}\frac{(g(x+h)-g(x))}{h} f(x)$$

Proof of $(fg)' = f'g + g'f$ II

$$= \lim_{h\to 0} \frac{(f(x+h) - f(x))g(x+h)}{h} + \lim_{h\to 0} \frac{f(x)(g(x+h) - g(x))}{h}$$

$$= \lim_{h\to 0} \frac{(f(x+h) - f(x))}{h} \lim_{h\to 0} g(x+h)$$

$$+ \lim_{h\to 0} \frac{(g(x+h) - g(x))}{h} f(x)$$

$$= f'(x)g(x) + g'(x)f(x).$$