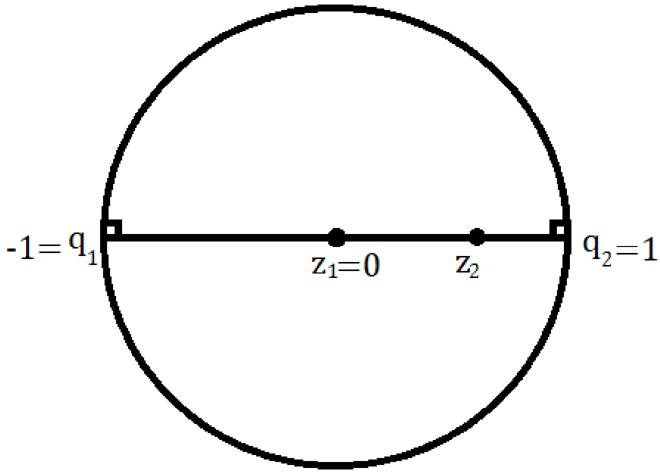
THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MMAT5120 Topics in Geometry 2018/19
Homework 2 Solutions

Solutions:

1. As cross ratio is invariant under Möbius transformations (and hence hyperbolic transformations), both left hand side and right hand side of the equation that we need to prove are invariant under hyperbolic transformations. So we can assume that $q_1 = -1, q_2 = 1, z_1 = 0$, and z_2 is a positive real number (by applying a suitable hyperbolic transformation), as shown as the figure below:



In that case, we have

$$\begin{aligned}\ln(z_1, z_2, q_2, q_1) &= \ln(0, z_2, 1, -1)\\ &= \ln(\frac{0-1}{0+1}\frac{z_2+1}{z_2-1})\\ &= \ln(\frac{1+z_2}{1-z_2})\\ &= d(0, z_2)\\ &= d(z_1, z_2)\end{aligned}$$

(Alternative solution: Using the upper half plane model and let $q_1 = 0, z_1 = i, z_2 = ki$ $(k > 1), q_2 = \infty$ is also a nice way of doing it)

2. We use the disk model, and put the centre of the hyperbolic circle at the origin. Then the hyperbolic circle is represented by a Euclidean circle with radius $r = \frac{e^R - 1}{e^R + 1} =$

$\tanh\frac{R}{2}$.
Parametrize the circle by Euclidean length, going from 0 to $2\pi r = 2\pi\tanh\frac{R}{2}$.

$$\begin{aligned}C =& 2\int_0^{2\pi\tanh\frac{R}{2}} \frac{1}{1-\tanh^2\frac{R}{2}}dt \\ =&(2)(2\pi\tanh\frac{R}{2})(\cosh^2\frac{R}{2}) \\ =&4\pi\sinh\frac{R}{2}\cosh\frac{R}{2} \\ =&2\pi\sinh R\end{aligned}$$

3. Let C_1, C_2 be two horocycles.
Using the upper half plane model and putting the ideal points of the horocycles at ∞, we know that C_1, C_2 are congruent to the $y = k_1$ and $y = k_2$ respectively, where k_1, k_2 are positive real numbers. (If you don't understand this step, please refer to the solution of classwork 3)
By considering the hyperbolic transformation $T(z) = \frac{k_2}{k_1}z$ on the upper half plane, we know that $y = k_1$ and $y = k_2$ are congruent . So we're done.

4. (a)

$$\begin{aligned}(1+i-j)(3k+2i) =& 3k+2i+3ik+2i^2-3jk-2ji \\ =&3k+2i-3j-2-3i+2k \\ =&-2-i-3j+5k\end{aligned}$$

(b) By the result of question 5 part (c), (which we will do later)

$$\begin{aligned}(1+i-j)^{-1} =& \frac{1-i+j}{1^2+1^2+1^2} \\ =&\frac{1}{3}-\frac{1}{3}i+\frac{1}{3}j\end{aligned}$$

5. (a) Consider the isomorphism $t+xi+yj+zk \leftrightarrow \begin{pmatrix} t & y & x & -z \\ -y & t & z & x \\ -x & -z & t & y \\ z & -x & -y & t \end{pmatrix}$.

$t-xi-yj-zk$ is represented by $\begin{pmatrix} t & -y & -x & z \\ y & t & -z & -x \\ x & z & t & -y \\ -z & x & y & t \end{pmatrix}$, which is the transpose of the matrix representing $t+xi+yj+zk$.
Hence $(qr)^* = r^*q^*$ follows from $(AB)^T = B^TA^T$ for matrices.

(Alternative solution: Letting $q = t_1+x_1i+y_1j+z_1k$, $r = t_2+x_2i+y_2j+z_2k$ and then expanding everything also works)

(b)

$$
\begin{aligned}
|qr|^2 =& (qr)(qr)^* \\
=& qrr^*q^* \text{ (by part (a))} \\
=& q|r|^2q^* \\
=& |r|^2qq^* \text{ (as } |r|^2 \in \mathbb{R}) \\
=& |r|^2|q|^2 \\
=& |q|^2|r|^2 \text{ (as } |q|^2, |r|^2 \in \mathbb{R}) \\
& \ldots \\
=& |rq|^2
\end{aligned}
$$

(c) For $q \neq 0$,
$q\frac{q^*}{|q|^2} = \frac{|q|^2}{|q|^2} = 1$
Hence $q^{-1} = \frac{q^*}{|q|^2}$
(there is no need check for $\frac{q^*}{|q|^2}q = 1$, because we know that left inverse and right inverse are the same for square matrices)

6.

$$
\begin{aligned}
qr = -rq \Leftrightarrow & -(q \cdot r) + (q \times r) = -(-(r \cdot q) + (r \times q)) \\
\Leftrightarrow & q \cdot r = -r \cdot q \text{ and } q \times r = -r \times q \\
\Leftrightarrow & q \cdot r = 0 \\
\Leftrightarrow & q \perp r
\end{aligned}
$$