THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MMAT5120 Topics in Geometry 2018/19
Classwork 2 Questions and solutions

Questions:

1. Let $Tz = \frac{z}{(1+i)z+i}$ be a Möbius transformation.
   (a) Find the fixed point(s) of T.
   (b) Find the normal form of T.
   (c) Draw the Steiner circles of the first and second kind with respect to the fixed points of T, and use "arrows" to indicate the action of T.

Solutions:

1. (a) Fixed points of T are points that satisfy $z = \frac{z}{(1+i)z+i}$.
   $z = 0$ is a fixed point.
   Note that $z = \infty$ is **not** a fixed point. $T(\infty) = \frac{1}{1+i} \neq \infty$.
   When $z \neq 0, \infty$, the equation reduces to $1 = \frac{1}{(1+i)z+i}$, which we can solve to obtain $z = \frac{1-i}{1+i} = -i$.
   So the fixed points are $0, -i$.

   (b) The normal form is $\lambda \frac{z}{z+i} = \frac{Tz}{Tz+i}$ for some $\lambda \in \mathbb{C}$. (or $\lambda \frac{z+i}{z} = \frac{Tz+i}{Tz}$ if you like)
   There are several ways to find λ. The obvious ways are simplifying $\lambda = \frac{z+i}{z}\frac{Tz}{Tz+i}$, or computing $R = STS^{-1}$ explicitly ($R(w) = STS^{-1}(w) = ST(\frac{-iw}{w-1}) = S(\frac{-iw}{w-i}) = -iw$. So $\lambda = -i$).
   Since $\lambda \frac{z}{z+i} = \frac{Tz}{Tz+i}$ holds for all values of z, two students did it in the clever/lazy way: Finding λ by substituting some nice values of z. We will do it here as well.
   When $z = -1$, $Tz = 1$, $\lambda \frac{-1}{-1+i} = \frac{1}{1+i}$. So $\lambda = \frac{1-i}{1+i} = -i$
   So the normal form is $-i\frac{z}{z+i} = \frac{Tz}{Tz+i}$
   (if your normal form is $\lambda \frac{z+i}{z} = \frac{Tz+i}{Tz}$, you should find $\lambda = i$)

   (c)