Quantum Cohomology (ref: Cox-Katz Ch 8 Quantum Coh.)

Recall cohomology ring $\cup : H^\bullet(X)\otimes H^\bullet(X)\longrightarrow H^\bullet(X)$

$$[Z]\cup[Z_2]=[Z_1'\pitchfork Z_2'] \quad \text{intersection product}$$

$$a\cup[X]=a$$

$\rightsquigarrow$ graded comm. ring w/ identity element. $1=[X]$

Reformulation using $H^n(X^n)\xrightarrow[\sim]{\int}\mathbb{Q}$

$\langle\ \ \rangle : H^\bullet(X)\otimes H^\bullet(X)\longrightarrow\mathbb{Q}$ non-degen. inner product. (Poincaré duality)

$$\langle a,b\rangle=\int a\cup b$$

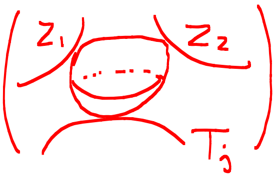
$$\Rightarrow\quad \langle[Z_1]\cup[Z_2],[Z_3]\rangle=\#(Z_1'\pitchfork Z_2'\pitchfork Z_3')$$

choose dual base T_j's and T^j's of $(H^\bullet(X),\langle\ \rangle)$, then

$$[Z_1]\cup[Z_2]=\#(Z_1\cap Z_2\cap T_j)\ T^j$$

Def: Small quantum product $a,b\in H^\bullet(X)$

$$a*b=\sum_j\sum_\beta\langle I_{0,3,\beta}\rangle(a,b,T_j)T^j\ \underbrace{e^{-\int_\beta\omega}}_{q^\beta}$$

$\#\Big($ 

$\Big)$

(holo. curve $\Rightarrow -\int_\beta\omega\le 0\Rightarrow q^\beta\le 1$
"$=$" $\Leftrightarrow\beta=0$, const. map)

Conj: converge if $\mathrm{Im}(\omega)\gg 0$ (large vol. limit/LVL)

- X Fano $\Rightarrow$ finite sum

Pf: $\mathrm{v.dim}_{\mathbb{C}}\overline{\mathcal{M}}_{0,n}(X,\beta)=\dim_{\mathbb{C}}X+\underbrace{\int_\beta c_1(X)}_{>0}+(n-3)$ ($\because$ Fano)

$\overset{?}{\|}$

$\deg a+\deg b+\deg T_j$ fixed.

Fix $N\Rightarrow\#\{\beta\text{ effective},\ n\in\mathbb{N}\mid \int_\beta c_1(X)+n=N\}<\infty$ even for big $QH^\bullet$.

($\int_\beta c_1(X)\ge 0$)

- CY $\Rightarrow\deg(a*b)=\deg a+\deg b\Rightarrow(\bigoplus_p H^{p,p},*)$ subring.

- $\int_X a*b = \int_X a\cup b = \langle a, b\rangle$ (i.e. $\nexists$ correction to $\langle\ \ \rangle$)

$\langle a*b, c\rangle = \langle a, b*c\rangle = \sum_\beta \langle I_{0,3,\beta}\rangle(a,b,c)\, q^\beta$

$=: \langle a, b, c\rangle$ 3-pt. fu. / A-correlat^{n} fu.

i.e. $a*b = \sum_i \langle a, b, T_i\rangle T^i$

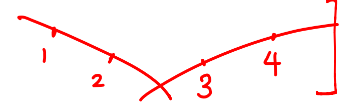
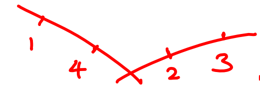
$= \int a*b*c$ $\left(\because \int a*b*c = \int (a*b)*c = \int (a*b)\cup c\right)$

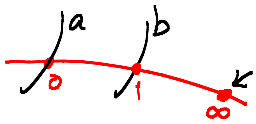
Theorem (Associativity / WDVV eqt).

$$(a*b)*c = a*(b*c)$$

Pf. $\langle (a*b)*c, d\rangle \overset{?}{=} \langle a*(b*c), d\rangle$

$\langle a*b, c*d\rangle \overset{?}{=} \langle a*d, b*c\rangle$

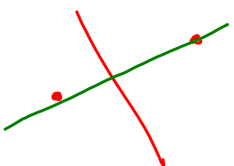
$\Big[$  $\Big] \overset{?}{=} \Big[$  $\Big] \in H^\bullet(\overline{\mathcal{M}}_{0,4}) = \mathbb{P}^1$.

[Note: $a*b =^{\mathbb{P}^1}$  cycle swipe out by such points]

i.e. $(H^\bullet(X), *)$ comm. ring w/ $1 = [X]$.

$(H^\bullet(X), *, \langle\ \ \rangle)$ Frobenius alg.

Eg. $QH^*_{sm}(\mathbb{P}^n) = \mathbb{C}[H]/H^{n+1} - q$.



$\Rightarrow \langle I_{0,3,1}\rangle(H, H^n, H^n) = 1$.

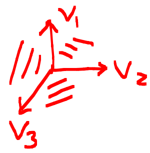
$\dim \mathcal{M} \Rightarrow \langle I_{0,3,d}\rangle(H^i H^j H^k) \neq 0 \Rightarrow i+j+k = r + d(r+1)$

$\Rightarrow d = \underset{\text{classical}}{0},\ \underset{\text{line}}{1}$

$\Rightarrow H * H^r = \underbrace{\langle I_{0,3,1}\rangle(H, H^n, H^n)}_{1}\, q H^0$ #

Eg. Toric variety X_Σ (smooth)

Σ fan w/ rays v_j's $\in N$



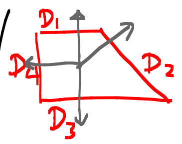
$\sim$ T-inv. divisors D_j's $\subset X_\Sigma$ w/ $j=1,\dots,s$

$$\left(X_\Sigma = \mathbb{C}^s /\!/ K_{\mathbb{C}^\times} \qquad 0 \to \underset{\substack{\wr\!\shortparallel \\ H_2(X,\mathbb{Z})}}{K} \to \mathbb{C}^s \xrightarrow{\vec{v}} N \to 0 \right)$$

$$H^2(X_\Sigma) \overset{\text{v.s.}}{=} \mathbb{C}\langle \overset{[D_1]}{\overset{\shortparallel}{x_1}}, \dots, \overset{[D_s]}{\overset{\shortparallel}{x_s}} \rangle / P(\Sigma)$$

w/ $P(\Sigma) \ni \sum_{i=1}^{s} \langle m, v_i \rangle x_i \quad \forall\ m \in M = N^*$

$$H^\bullet(X_\Sigma) \overset{\text{ring}}{=} \mathbb{C}[x_1,\dots,x_s] / P(\Sigma) + SR(\Sigma)$$



$D_1 \cdot D_3 = 0 = D_2 \cdot D_4$
$\rightsquigarrow$ SR relation

$SR(\Sigma) \ni x_{i_1} \cdots x_{i_k}$ if $P = \{v_{i_1},\dots,v_{i_k}\} \not\subset \sigma \quad \forall\ \sigma \in \Sigma$

Stanley-Reisner ideal

$\Longleftrightarrow\ \ni\ x_{i_1} \cdots x_{i_k}$ $\begin{cases} P = \{v_{i_1},\dots,v_{i_k}\} \not\subset \sigma \quad \forall\ \sigma \in \Sigma \\ \text{but every subset does !} \end{cases}$

primitive collection.

$$v_P := v_{i_1} + \cdots + v_{i_k} \in \sigma = \mathrm{Convex}(v_{j_1},\dots,v_{j_\ell})$$

$$= c_1 v_{j_1} + \cdots + c_\ell v_{j_\ell}$$

$$\rightsquigarrow \quad v_{i_1} + \cdots + v_{i_k} - c_1 v_{j_1} - \cdots - c_\ell v_{j_\ell} = 0$$

$$\rightsquigarrow \quad \beta(P) \in H_2(X,\mathbb{Z}) = \mathrm{Ker}\left(\mathbb{Z}^s \overset{\vec{v}}{\twoheadrightarrow} N \right)$$

Batyrev: $\beta(P)$ effective $\rightsquigarrow$ should contribute to $QH^\bullet$

Batyrev ring $H^\bullet_\omega(X_\Sigma) \triangleq \mathbb{C}[x_1,\dots,x_s]/P(\Sigma) + SR_\omega(\Sigma)$

$$SR_\omega(\Sigma) \ni x_{i_1} \cdots x_{i_k} - q^{\beta(P)} x_{j_1}^{c_1} \cdots x_{j_\ell}^{c_\ell}, \quad P: \text{prim. coll.}$$

$(= QH^\bullet(X_\Sigma)$ if Fano$)$.

• V CY^3

(i) $a * b = a \cup b$ unless $a, b \in H^2$

$(\because \dim \overline{\mathcal{M}}_{g,0}(V, \beta) = (\dim V - 3)(g-1) + \int_\beta c_1(X) = 0 \quad \forall \beta)$.

(ii) For $a, b, c \in H^2$,

$$\langle a,b,c\rangle = \int_V a*b*c = \sum_\beta \langle I_{0,3,\beta}\rangle(a,b,c)\, q^\beta$$

$$= \int_V a \cup b \cup c + \sum_{\beta \neq 0} n_\beta \frac{q^\beta}{1-q^\beta} \int_\beta a \int_\beta b \int_\beta c$$

where $\langle I_{0,0,\beta}\rangle = \sum_{\beta = k\gamma} \frac{n_\gamma}{k^3}$ ($\sim$ multiple cover formula)

(iii) simple flops change $H^\cdot(X)$

but NOT $QH^\cdot(X)$

• Coeff. : Novikov ring

$$\Lambda(\omega, Q) \ni \sum_{\beta \in H_2(M,\mathbb{Z})} \underset{\mathbb{Q}}{\overset{}{a_\beta}}\, q^\beta \quad \text{s.t. } \#\left\{\beta \,\middle|\, \begin{array}{l} a_\beta \neq 0 \\ \int_\beta \omega < c \end{array}\right\} < \infty$$

($\sim$ compactness if bdd energy $\int |du|^2 \overset{\bar\partial u = 0}{=\!=} \int_\beta u^*\omega$)

• Big $QH^\cdot$

i.e. $\underset{\text{Big}}{\langle I_{0,n,\beta}\rangle}$ vs $\underset{\text{small}}{\langle I_{0,3,\beta}\rangle}$

GW potential $\Phi : H^\cdot(X) \longrightarrow \Lambda$

$$\Phi(\gamma) = \sum_{n=0}^\infty \sum_{\beta \in H_2} \frac{1}{n!} \langle I_{0,n,\beta}\rangle(\gamma^n)\, q^\beta$$

$$T_i \underset{\text{big}}{*} T_j \triangleq \sum_k \underbrace{\frac{\partial^3 \Phi}{\partial t_i \partial t_j \partial t_k}}_{\sum_{n,\beta} \frac{1}{n!}\langle I_{0,n+3,\beta}\rangle(T_i\, T_j\, T_k\, \gamma^n)\, q^n} T^k$$ depending on $\gamma \in H^\cdot(X)$

• $*_{\text{big}}$ associative (same proof) ✓

(i.e. WDVV eqt. : $\sum_a \Phi_{t_i t_j t_a} \Phi_{t^a t_k t_\ell} = (\pm) \sum_a \Phi_{t_j t_k t_a} \Phi_{t^a t_i t_\ell}$

Eg. $\Phi_{\mathbb{P}^2} = \underbrace{\frac{1}{2}(t_0^2 t_2 + t_0 t_1^2)}_{\text{Classical}} + \sum_{d=1}^{\infty} N_d e^{dt_1} \frac{t_2^{3d-1}}{(3d-1)!} q^d$

$T_1 * T_1 = T_2 + \Phi_{111} T_1 + \Phi_{112} T_0$

Eg. $\Phi_{\text{QuinticCY3}} = \underbrace{\frac{1}{2} t_0^2 t_3 + t_0 t_1 t_2 + \frac{5}{6} t_1^3 - \sum_{i<j} g_{ij} t_0 \overset{H^3}{\hat{u}_i \hat{u}_j}}_{\text{classical}} + \sum_{d \geq 1} \langle I_{0,0,d} \rangle (e^{t_1} q)^d$

$T_1 * T_1 = \Phi_{111} T_2$, $\quad QH_{big} \sim QH_{small}$

- Dubrovin formalism

inner vector space $(M, \langle\ \rangle) \xrightarrow{F} \mathbb{C}$

1° $\nabla^3 F = (A_{ijk}) \in \Gamma(M, S^3 T_M^*)$

$\qquad (A_{ij}^k) \qquad \Gamma(M, \mathrm{Hom}(S^2 T_M, T_M))$ ($\cap$ via $\langle\ \rangle$)

$\rightsquigarrow$ (i) (comm). $T_i * T_j = \sum_k A_{ij}^k T_k$

(ii) (assoc.) $\iff$ WDVV eqt.

2° $A = \nabla^3 F \in \Omega^1(M, \mathrm{End}\, T_M)$

$\rightsquigarrow$ Connection on T_M : $\nabla^\lambda = d + \lambda A$

Dubrovin connection.

(i) Torsion free

(ii) Flat ($dA = 0 = A^2$) $\iff$ WDVV eqt.

$T_0 = $ id. for $*$ $\iff \nabla^\lambda_{\frac{\partial}{\partial t_0}} \left(\frac{\partial}{\partial t_i}\right) = \lambda \frac{\partial}{\partial t_i} \quad \forall i$

$\iff A_{0ij} = g_{ij}$

$\rightsquigarrow$ $(M, \langle\ \rangle, *)$ Frobenius mfd.

- ∇^λ w/ $\lambda \in \mathbb{C} \subset \mathbb{C} \cup \infty = \mathbb{P}^1$

extends to $M \times \mathbb{P}^1$ w/ regular sing. pt. at $\lambda = 0, \infty$.

($\sim$ Painlevé VI via Laplace transform).

- A - VHS (Variation of Hodge Structure in A-side).

$$H^\bullet(X) = H^0 + H^2 + H^{>2}$$
$$\gamma = \underbrace{t_0 T_0}_{\varepsilon} + \underbrace{\sum_{i=1}^{r} t_i T_i}_{\delta} + \underbrace{\sum_{i=r+1}^{m} t_i T_i}_{\varepsilon}$$

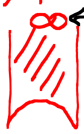
$$T_i * T_j = \sum_{k,n,\beta} \frac{1}{n!} \langle I_{0,n+3,\beta} \rangle (T_i, T_j, T_k, \varepsilon^n)\, e^{\int_\beta \delta}\, q^\beta\, T^k$$

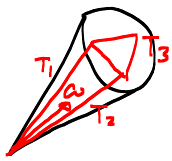
$H^\cdot$ basis (k), markpt. (n), rep. class (β) (∵ divisor axiom)

- $*_{big}|_{\gamma=0} = *_{small}$

- Complexified Kähler cone

$$K_{\mathbb{C}}(V) = \{\omega \in H^2_{\mathbb{C}}(V) \mid \mathrm{Im}\,\omega > 0\} / H^2_{\mathbb{Z}}(V)$$

(e.g. elliptic curve  ∞ Large Volume Limit (LVL) $\overset{\sim}{\rightarrow}$  $\mathbb{P}^1$.

Choose $T_1, \ldots, T_r \in \overline{K(V)_{\mathbb{Z}}}$  $\xrightarrow{\text{restr.}} K_{\mathbb{C}}(V)$

$\omega = \sum_{i=1}^{r} u_i T_i$

$q^\beta = e^{2\pi i \int_\beta \omega}$ → good (∵ $T_i/\mathbb{Z}$) (assume cgce for GW).

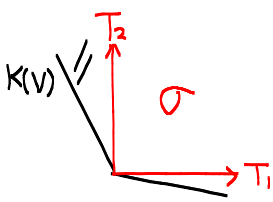
⟿ flat connection ∇ on $\mathcal{H} = H^\bullet \times K_{\mathbb{C}}(V)$, $\mathcal{H} \to K_{\mathbb{C}}(V)$ (via $*_{sm}$)

(A-model connection)

LVL : $\sigma :=$ simplicial cone gen. by $T_1, \ldots, T_r \subseteq K(V)$

$$\sum^r u_i T_i \qquad D_\sigma := \frac{H^2(V,\mathbb{R}) + i\sigma}{H^2(V,\mathbb{Z})} \subset K_{\mathbb{C}}(V)$$

$$\big(\underbrace{e^{2\pi i u_1}}_{q_1}, \ldots, \underbrace{e^{2\pi i u_r}}_{q_r}\big) \quad (\Delta^*)^r \subset \Delta^r \ni 0 \quad \text{LVL}$$



$\Rightarrow D_\sigma = \infty \times \infty \simeq (\Delta^*)^2 \subset \Delta^2$ (D_1, D_2, LVL.)

Prop: ∇ reg. sing. pt. along $\Delta^r - D_\sigma = \bigcup_{j=1}^{r} D_j$

T_j (monodromy about D_j) unipotent

$N_j := \log T_j \underset{\text{conj.}}{\sim} -T_j \cup (-\cdot-)$

Pf: $q_j = e^{u_j}$ (up to $2\pi i$) $\Rightarrow \frac{\partial}{\partial u_j} = q_j \frac{\partial}{\partial q_j}$

$$\nabla_{\frac{\partial}{\partial q_j}} T_k = \frac{1}{q_j} \sum_{l,\beta} \langle I_{0,3,\beta} \rangle (T_j\, T_k\, T_l) T^l q^\beta$$

$q^\beta = q_1^{\int_\beta T_1} \cdots q_r^{\int_\beta T_r}$, $\int_\beta T_r \geq 0$ ($\because T_i \in$ Kähler cone)

$\rightsquigarrow$ at worst log pole (reg. sing. pt.).

$\Downarrow$

$q^\beta \to 0$ as $q_j \to 0$ if β effective

$\Rightarrow$ Residue matrix of $\nabla_{\frac{\partial}{\partial q_j}}$ at $q_j = 0$

$$\mathrm{Res}_{q_j=0}(\nabla) = \left(\int_V T_j \cup T_k \cup T_l \right)_{l,k}$$

$=$ Matrix for $T_j \cup (-)$ $\leftarrow$ nilpotent

$\Rightarrow$ e.v. of $\mathrm{Res}_{q_j=0}(\nabla)$: zero.

"$\Rightarrow$" $T_j \underset{\text{conj.}}{\sim} e^{-2\pi i \mathrm{Res}_{q_j=0}(\nabla)}$ #

$\rightsquigarrow$ $\mathcal{H}, \nabla$ extends $\tilde{\mathcal{H}}$, $\nabla^c = \nabla + \sum N_j\, du_j$

$\mathcal{H} \downarrow (\Delta^\times)^r$ $\qquad$ $\tilde{\mathcal{H}} \downarrow \Delta^r$

$\nabla s = 0$, s : multi-value $\rightsquigarrow$ $\tilde{s} := (e^{-\sum u_j N_j}) \cdot s$ sing. value for $\tilde{\mathcal{H}}$.

$\nabla^c \tilde{s} = 0$

Prop: $\forall\, T_k \quad \exists!\ \nabla^c \tilde{s}_k = 0$ s.t.

$$\begin{cases} \tilde{s}_k = T_k + \text{h.o.t.} \\ \tilde{s}_k(0) = T_k \end{cases}$$