Bezrukavnikov + Kaledin. Fedorsov quantization in algebraic context (paper).

• Deformation quantization problem:

(Sheaf of) [smooth/holomorphic] functions on $\overset{\text{sympl.}}{(M,\omega)}$

$\exists ?$ commutative $\xrightarrow{\text{deform}}$ non-comm. alg. ?

(1) Linear (V,ω), $S^\bullet V^* = \mathbb{C}[V] \rightsquigarrow$ Weyl alg. ✓

(formal polydisc) $\mathcal{a} = \mathbb{C}[[x_1,\dots,x_d,y_1,\dots,y_d]]$

$\rightsquigarrow D = \mathcal{a}[[\hbar]]$ s.t. $[x_i,y_i]=\hbar$, other $[\ ]=0$.

$$\Rightarrow \quad \frac{1}{\hbar}D \longrightarrow \mathrm{Der}(D)$$

Indeed, we have exact seq. of Lie algebras:

$$0 \longrightarrow \mathbb{C}[[\hbar]] \longrightarrow \frac{1}{\hbar}D \longrightarrow \mathrm{Der}(D) \longrightarrow 0$$

(2) C^∞ (Fedorsov) (3) Holom/alg. (This paper).

§ 1. X smooth scheme of finite type (or family/S)

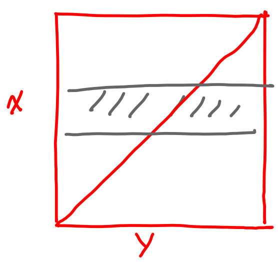
$\rightsquigarrow$ deRham cpx. $\Omega^\bullet_X$ $f^\cdot$

$\xrightarrow{\text{hypercohom.}}$, $H^\bullet_{DR}(X)$ $(\simeq H^\bullet_{sing}(X,\mathbb{C}))$

• $\mathcal{E}$: VB/X $\rightsquigarrow$ jet bundle $J^\infty\mathcal{E} = \pi_{1*}\pi_2^*\mathcal{E}$

($X\times X \supset \Delta \rightsquigarrow$ completion along diag, $\hat{\Delta}$

$$\begin{tikzcd} & \hat{\Delta} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X & & X \end{tikzcd}$$



$f(y) = f(x) + f'(x)(y-x) + \frac{f''(x)}{2}(y-x)^2 + \dots$

$(f(x), f'(x), f''(x), \dots) \in J^\infty\mathcal{O}_X$

Connection: $\nabla_{\frac{\partial}{\partial x}}\left(f(x), f^{(1)}(x), f^{(2)}(x), \dots\right)$

$$:= \left(0, \frac{\partial f}{\partial x} - f^{(1)}, \frac{\partial f^{(1)}}{\partial x} - f^{(2)}(x), \dots\right)$$

$J^\infty \mathcal{E}$: flat bundle / X

(flatness of ∇ ~ indep. diff. wrt x_1 and x_2)

(local) flat sections of $J^\infty\mathcal{E}$ $\longleftrightarrow$ holo. sections of $\mathcal{E}$

(i.e. $\nabla(s_0, s_1, s_2, s_3, \dots) = 0$

$\iff$ $s_0 + s_1 + s_2 + s_3 + \dots$ is Taylor expansion.

$$H^\bullet_{DR}(X, \underbrace{J^\infty\mathcal{E}}_{\text{flat bdl.}}) = H^\bullet(X, \underbrace{\mathcal{E}}_{\text{coh. shf.}})_{\text{Dol. cohom.}}$$

$$0 \longrightarrow \Omega_X^{\geq 1} \longrightarrow \Omega_X^\bullet \longrightarrow \Omega_X^0 = \mathcal{O}_X \longrightarrow 0$$

$$\rightsquigarrow \cdots \longrightarrow \underbrace{H^i_F(X)}_{\text{cohom.}} \longrightarrow H^i_{DR}(X) \longrightarrow H^i(X, \mathcal{O}_X) \longrightarrow H^{i+1}_F(X) \longrightarrow \cdots$$

Def: X admissible

$\overset{\Delta}{\iff}$ $H^i_{DR}(X) \xrightarrow[\text{onto}]{} H^i(X, \mathcal{O}_X)$, $i = 1, 2$.

$\iff$ $0 \longrightarrow H^2_F(X) \longrightarrow H^2_{DR}(X) \longrightarrow H^2(X, \mathcal{O}_X) \longrightarrow 0$

e.g. projective mfd ($\because$ Hodge theory).

Def: Quantization

$\mathcal{D}$: shf of assoc. flat $\mathcal{O}_X[\![\hbar]\!]$ on X,
complete in $\hbar$-adic topology,
$\mathcal{D}/\hbar\mathcal{D} \simeq \mathcal{O}_X$.

$a, b \in \mathcal{O}_X$, choose any lift $\tilde{a}, \tilde{b} \in \mathcal{D}$

$\Rightarrow \{a, b\} := \frac{1}{\hbar}(\tilde{a}\tilde{b} - \tilde{b}\tilde{a}) \pmod{\hbar^2}$ well-def^{d}.

$\rightsquigarrow (\mathcal{O}_X, \{\ \})$ Poisson scheme.

(we are in non-degen. case, namely symplectic).

Lemma (Darboux type).

$\mathcal{D}$: non-degen. quantizat^{n} of formal polydisc.

$\Rightarrow \mathcal{D} \simeq$ formal Weyl alg.

Theorem: (X^{2d}, Ω) admissible, symplectic.

$\exists$ natural non-comm. period map

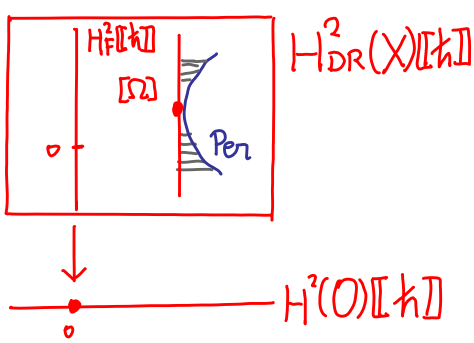
$$\mathrm{Per} : \underbrace{\{\text{quant. of } (X, \Omega)\}/\simeq}_{Q(X,\Omega)} \hookrightarrow H^2_{DR}(X)[\![\hbar]\!]$$

s.t. $\mathrm{Per}(q) = [\Omega] + O(\hbar)$

$Q(X,\Omega) \xhookrightarrow{\mathrm{Per}} H^2_{DR}(X)[\![\hbar]\!] \underset{P}{\leftrightarrows} H^2_F(X)[\![\hbar]\!] \rightsquigarrow$ isom.

$\forall$ splitting

($\mathcal{D}$ canonical if $\mathrm{Per}(q) = [\Omega]$).



Say X cpt holo. sympl., then

$$H^2_F = H^0(\Lambda^2 T^*) + H^1(T^*)$$
$$\overset{\lrcorner\Omega}{=} H^0(\Lambda^2 T) + H^1(T)$$

n.c. deform. comm. deform.

$H^2(\mathcal{O}) \leftarrow$ gerbe direction

§2. Def. Harish-Chandra pair $\langle G, \mathfrak{h} \rangle$

conn. affine alg. group $G \quad \mathfrak{g} \underset{\substack{\shortparallel \\ d(\text{adj. action})}}{\hookrightarrow} \mathfrak{h}$ Lie alg. emb.

Def. HC mod $G \curvearrowright V \curvearrowleft \mathfrak{h}$ compatible.

$$\begin{array}{l} \mathcal{M} \\ p \downarrow \\ X \end{array} \quad \begin{array}{l} G\text{-torsor} \rightsquigarrow \text{Lie alg. bdl}/X : \mathfrak{g}_{\mathcal{M}} \to \mathfrak{h}_{\mathcal{M}} \\ (\text{anolog} \sim \text{Principal } G\text{-bundle}) \end{array}$$

$$0 \to \underbrace{T_{\mathcal{M}/X}}_{T_{\mathrm{vert}}\mathcal{M}} \to T_{\mathcal{M}} \to p^* T_X \to 0 \qquad /\mathcal{M}$$

Recall: Atiyah class.

$$\mathbb{C}^r \longrightarrow E \longrightarrow X \qquad \text{any holo. VB.}$$

any connection D w/ $D^{0,1} = \bar{\partial}_E$

$$\Longrightarrow \quad \bar{\partial}_E F^{1,1} = 0$$

$$\Longrightarrow At(E) \triangleq [F^{1,1}] \in H^{1,1}_{\bar{\partial}}(X, \mathrm{End}\, E) \simeq \mathrm{Ext}^1_{\mathcal{O}_X}(T_X, \mathrm{End}\, E)$$

$\rightsquigarrow$ extension (indep. of choice of D)

$$0 \longrightarrow \mathrm{End}\, E \longrightarrow \mathcal{E} \longrightarrow T_X \longrightarrow 0$$

or $$0 \longrightarrow E \longrightarrow J^1(E) \longrightarrow T_X \otimes E \longrightarrow 0$$

Indeed $J^1(E)$ is the 1st jet bundle of E.

Similarly for principal G-bundle$/X$

$$G \longrightarrow P \xrightarrow{p} X$$

$$\rightsquigarrow 0 \longrightarrow P \times_G \mathfrak{g}^{\circlearrowleft \mathrm{Ad}} \longrightarrow \mathcal{E} \longrightarrow T_X \longrightarrow 0$$

θ connection

$$p^* \theta \in H^0(P, T_P^* \otimes \mathfrak{g}) \qquad G\text{-inv.}$$

Back to $\langle G, \mathfrak{h} \rangle$-torsor, $\mathcal{M} \longrightarrow X$

$$\rightsquigarrow \quad 0 \to \mathfrak{g}_{\mathcal{M}} \xrightarrow{i_{\mathcal{M}}} \mathcal{E}_{\mathcal{M}} \longrightarrow T_X \to 0 \qquad /X$$

G-inv. conn: $\mathfrak{g}_{\mathcal{M}} \downarrow \mathfrak{h}_{\mathcal{M}}$, $\theta_{\mathcal{M}}$ splitting, $\mathcal{E}_{\mathcal{M}} \to \mathfrak{h}_{\mathcal{M}}$

$\rightsquigarrow$ G-inv. $\quad p^* \theta_{\mathcal{M}} \in H^0(\mathcal{M}, \Omega^1_{\mathcal{M}} \otimes \mathfrak{g})$

Flat if $\quad 2\, d(p^* \theta_{\mathcal{M}}) + (p^* \theta_{\mathcal{M}})^2 = 0$

Def: HC-torsor : $\mathcal{M} \longrightarrow X$ is G-torsor
+ <u>flat</u> $\mathfrak{h}$-valued conn. $\theta_{\mathcal{M}} : \mathcal{E}_{\mathcal{M}} \longrightarrow \mathfrak{h}_{\mathcal{M}}$

Call <u>transitive</u> if $\theta_{\mathcal{M}} \cong$

Write: $\{$ HC-torsor$/X \}/\cong \;=: H^1(X, \langle G, \mathfrak{h} \rangle)$

(analog: $\{$ $\overset{\text{flat}}{\text{principal}}$ G-bdl$/X\}/\cong \;= H^1(X, G^{\delta})$)

- $\langle \mathcal{M}, \theta_{\mathcal{M}} \rangle \longrightarrow X$: $\langle G, \mathfrak{h} \rangle$-torsor

  V : finite dim $\langle G, \mathfrak{h} \rangle$-mod.

  $\rightsquigarrow f : \langle G, \mathfrak{h} \rangle \longrightarrow \langle GL(V), \mathfrak{gl}(V) \rangle$

  $\rightsquigarrow f_* \mathcal{M}$ induced torsor $/X$

  $\mathrm{Loc}(\mathcal{M}, V) = \mathcal{V}$ flat VB$/X$ (frame bdl.)

$$\mathrm{Loc}(\mathcal{M}, -) : \underbrace{H^{\bullet}(\langle G, \mathfrak{h} \rangle, V)}_{\substack{\text{Lie alg. cohom.} \\ \text{(w/ valued in } V)}} \longrightarrow H^{\bullet}_{DR}(X, \mathcal{V}).$$

Analog: Principal (flat) G-bdl P/X + $G \xrightarrow{f} GL(V)$
$\rightsquigarrow$ associated (flat) VB $\quad V \to \mathcal{V} = P \overset{\times}{_G} V \longrightarrow X$
assoc. frame bdl. $\quad P \overset{\times}{_G} \mathfrak{g}_{Ad} \longrightarrow X$

Lemma 2.6 $\langle G,\mathfrak{h}\rangle \to \mathcal{M} \to X$ transitive HC-torsor

$$(\!(\text{flat VB}/X)\!) \simeq (\!(\langle G,\mathfrak{h}\rangle\text{-equivar. VB}/\mathcal{M})\!)$$

- V : $\langle G,\mathfrak{h}\rangle$-mod

$\rightsquigarrow$ exact sequence of HC-pairs (~semi-direct product)

$$1 \longrightarrow \langle V,V\rangle \xrightarrow{\rho} \langle G_1,\mathfrak{h}_1\rangle \xrightarrow{\pi} \langle G,\mathfrak{h}\rangle \to 1$$

- $\mathcal{M} \longrightarrow X$: $\langle G,\mathfrak{h}\rangle$-torsor

  V : $\langle G,\mathfrak{h}\rangle$-mod

$$H^1_{\mathcal{M}}(X,\langle G_1,\mathfrak{h}_1\rangle) \triangleq \left\{ \begin{array}{l} \langle G_1,\mathfrak{h}_1\rangle\text{-torsor } \mathcal{M}_1/X \\ \text{w/ } \pi_*\mathcal{M}_1 \simeq \mathcal{M} \end{array} \right\} /\cong$$

call lifting of $\mathcal{M}$.

i.e.
$$\begin{array}{ccc} H^1(X,\langle G_1,\mathfrak{h}_1\rangle) & \longrightarrow & H^1(X,\langle G,\mathfrak{h}\rangle) \\ \cup\!| & \square & \uplus \\ H^1_{\mathcal{M}}(X,\langle G_1,\mathfrak{h}_1\rangle) & \longrightarrow & \mathcal{M} \end{array}$$

Prop. 2.7 $\langle G,\mathfrak{h}\rangle \to \mathcal{M} \to X$ transitive, then

(i) $\exists\ c \in H^2(\langle G,\mathfrak{h}\rangle, V)$ s.t.

$$H^1_{\mathcal{M}}(X,\langle G_1,\mathfrak{h}_1\rangle) \neq \phi \iff \mathrm{Loc}(\mathcal{M},c) = 0 \in H^2_{DR}(X,\mathcal{V})$$

(ii) $\Rightarrow$ ↑ torsor over $H^1_{DR}(X,\mathcal{V})$.

[Roughly, '$\exists$' exact seq.

$$H^1_{DR}(X,\mathcal{V}) \to H^1(X,\langle G_1,\mathfrak{h}_1\rangle) \longrightarrow H^1(X,\langle G,\mathfrak{h}\rangle) \longrightarrow H^2_{DR}(X,\mathcal{V})$$

$$\mathcal{M} \longmapsto \mathrm{Loc}(\mathcal{M},c)$$

$$
\begin{tikzcd}
0 \arrow[r] & U \arrow[r] \arrow[d, equal] & V \arrow[r] \arrow[d] & W \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & U \arrow[r] & \langle G_1, h_1 \rangle \arrow[r] \arrow[d] & \langle G_0, h_0 \rangle \arrow[r] \arrow[d] & 0 \\
 & & \langle G, h \rangle \arrow[r, equal] & \langle G, h \rangle &
\end{tikzcd}
$$

as $\langle G, h\rangle$-mod

Given $\langle G, h \rangle \longrightarrow \mathcal{M} \longrightarrow X$

$\rightsquigarrow$ VB/X $\quad 0 \to \mathcal{U} \longrightarrow \mathcal{V} \longrightarrow \mathcal{W} \longrightarrow 0$

$\rightsquigarrow \quad H^2_{DR}(X, \mathcal{U}) \longrightarrow \underset{\ni}{H^2_{DR}(X, \mathcal{V})} \longrightarrow H^2_{DR}(X, \mathcal{W})$

If $0 \neq Loc(\mathcal{M}, c) \longmapsto 0$

i.e. $\quad H^1(X, \langle G_1, h_1 \rangle) \longrightarrow \underset{\ni}{H^1(X, \langle G_0, h_0 \rangle)}$

$\not\ni \cancel{X} \longrightarrow \exists\, \mathcal{M}_1$

Lemma 2.8. $\quad H^1_{\mathcal{M}}(X, \langle G_0, h_0 \rangle) \longrightarrow H^2_{DR}(X, \mathcal{U})$

$\mathcal{M}_1 \longmapsto Loc(\mathcal{M}_1, c_0)$

is compatible w/ $H^1_{DR}(X, \mathcal{W})$-action.

i.e.

$$
\begin{tikzcd}
H^1_{DR}(X,\mathcal{U}) \arrow[r] \arrow[d, equal] & H^1_{DR}(X,\mathcal{V}) \arrow[r] \arrow[d] & H^1_{DR}(X,\mathcal{W}) \arrow[r] \arrow[d] & H^2_{DR}(X,\mathcal{U}) \arrow[d, equal] \\
H^1_{DR}(X,\mathcal{U}) \arrow[r] & H^1(X,\langle G_1,h_1\rangle) \arrow[r] \arrow[d] & H^1(X,\langle G_0,h_0\rangle) \arrow[r, "\mathcal{M}_1 \mapsto Loc(\mathcal{M}_1, c_0)"] \arrow[d] & H^2_{DR}(X,\mathcal{U}) \\
 & H^1(X,\langle G,h\rangle) \arrow[r, equal] \arrow[d, "Loc(\mathcal{M}, c) \mapsto 0"'] & H^1(X,\langle G,h\rangle) \arrow[d] & \\
H^2_{DR}(X,\mathcal{U}) \arrow[r] & H^2_{DR}(X,\mathcal{V}) \arrow[r] & H^2_{DR}(X,\mathcal{W}) \arrow[r] & H^3_{DR}(X,\mathcal{U})
\end{tikzcd}
$$

§3. • $\mathcal{A} = \mathbb{C}[[x_1, \dots, x_n]]$

$$\begin{array}{lll} W = \mathrm{Der}(\mathcal{A}) & & = \{\text{v.f. on formal polydisc}\} \\ \cup\!\!\mathsf{I} & & \\ W_0 & & = \{\text{v.f., vanish at } 0\} \\ \| & & \\ \mathrm{Lie}\,\mathrm{Aut}(\mathcal{A}) & \rightsquigarrow \text{HC-pair} & \langle \mathrm{Aut}(\mathcal{A}), W \rangle \end{array}$$

• $X \rightsquigarrow$ scheme $\mathcal{M}_{coord}$ of formal coord. system. (non-Noeth)

$\langle \mathrm{Aut}\,\mathcal{A}, W \rangle \longrightarrow \mathcal{M}_{coord} \longrightarrow X$ transitive HC-torsor

(nonlinear analog of $\mathbb{R}^n \curvearrowleft GL(n, \mathbb{R}) \curvearrowright \{\text{bases of } \mathbb{R}^n\}$)

$W \curvearrowright \mathcal{A} \rightsquigarrow$ Atiyah bundle of $\mathcal{M}_{coord}$

$\rightsquigarrow \quad 0 \to W_{\mathcal{M}} \xrightarrow{a} \mathcal{E}_{\mathcal{M}} \to T_X \to 0 \qquad / X$

$\rightsquigarrow \quad \theta_{\mathcal{M}} := a^{-1}$ flat W-valued conn.

$\langle \mathrm{Aut}\,\mathcal{A}, W \rangle \longrightarrow \langle \mathcal{M}_{coord}, \theta_{\mathcal{M}} \rangle \longrightarrow X$

bdl. of formal coord. system on X.

• $\langle \mathrm{Aut}\,\mathcal{A}, W \rangle \curvearrowright \mathcal{A} \overset{\text{assoc. bdl.}}{\rightsquigarrow} J^{\infty}\mathcal{O}_X \xrightarrow{\text{flat sect}^{\text{n}}} \mathcal{O}_X$

• $\langle \mathrm{Aut}\,\mathcal{A}, W \rangle \curvearrowright W \overset{\text{assoc. bdl.}}{\rightsquigarrow} J^{\infty}T_X \rightsquigarrow T_X$

$\qquad \curvearrowright \Omega^p \mathcal{A} \rightsquigarrow J^{\infty}\Omega^p_X \rightsquigarrow \Omega^p_X .$

Now, symplectic case $\left\{\begin{array}{l} \mathcal{A} = \mathbb{C}[\![x_1, \dots, x_d, y_1, \dots, y_d]\!] \\ \omega = \sum dx_i \wedge dy_i . \end{array}\right.$

$W \supseteq H = \{\text{Hamil. v.f.}\}$

$W_0 \cap H = \operatorname{Lie} \operatorname{Sympl}(\mathcal{A})$.

$\rightsquigarrow$ HC-pair $\langle \operatorname{Sympl}(\mathcal{A}), H \rangle$.

Lemma 3.2 sympl. str. on X $\longleftrightarrow$ reduct$^{\text{n}}$ of $\langle \operatorname{Aut}\mathcal{A}, W\rangle$-torsor $\mathcal{M}_{\text{coord}}$ to $\langle \operatorname{Sympl}\mathcal{A}, H\rangle$-torsor. (denote $\mathcal{M}_s$)

i.e.

$$\begin{tikzcd} \langle \operatorname{Aut}\mathcal{A}, W\rangle \arrow[r] & \mathcal{M}_{\text{coord}} \arrow[r] & X \\ \langle \operatorname{Sympl}\mathcal{A}, H\rangle \arrow[r] \arrow[u, phantom, "\vee|"] & \mathcal{M}_s \arrow[r] \arrow[u, hook] & X \arrow[u, equal] \end{tikzcd}$$

Quantized version :

$\operatorname{Der} D \triangleq \{\mathbb{C}[\![\hbar]\!]\text{-linear derivations of } D\}$. $\quad D/\hbar D \cong \mathcal{A}$

$\rightsquigarrow$ $\operatorname{Der} D \xrightarrow{a} W$

- $\operatorname{Im}(a) = H \subset W$ (Ex).
- $a^{-1}(W^0) = \operatorname{Lie} \operatorname{Aut} D$

where $\operatorname{Aut}(D) = \{\text{autom. preserve. } m_{\mathcal{A}} + \hbar D \triangleleft D\}$

$\rightsquigarrow \langle \operatorname{Aut} D, \operatorname{Der} D\rangle \to \langle \operatorname{Sympl}\mathcal{A}, H\rangle$.

$Q(X, \Omega) := \{\text{quantizations of } X\}/\cong$

$$\begin{tikzcd} H^1(X, \langle \operatorname{Aut} D, \operatorname{Der} D\rangle) \arrow[r] & H^1(X, \langle \operatorname{Symp}\mathcal{A}, H\rangle) \\ H^1_{\mathcal{M}_s}(X, \langle \operatorname{Aut} D, \operatorname{Der} D\rangle) \arrow[r, mapsto] \arrow[u, phantom, "\cup"] \arrow[ur, phantom, "\square"] & \mathcal{M}_s \arrow[u, phantom, "\in" rotate=90] \end{tikzcd}$$

Lemma 3.4 (X^{2d}, Ω) sympl.. Then

$Q(X, \Omega) \leftrightarrow H^1_{\mathcal{M}_s}(X, \langle \operatorname{Aut} D, \operatorname{Der} D\rangle)$

Structure of $\mathrm{Der}\,\mathcal{D}$.

- $$0 \to \mathbb{C}[\![\hbar]\!] \to \overbrace{\hbar^{-1}\mathcal{D}}^{G} \to \mathrm{Der}\,\mathcal{D} \to 0$$ central ext^{n} of Lie alg.

  ($\because$ every $d \in \mathrm{Der}\,\mathcal{D}$ is commutator w/ $\exists \tilde{d} \in \hbar^{-1}\mathcal{D}$)

Let $(\mathrm{Der}\,\mathcal{D})_{>p} \triangleq \hbar^{p+1}\mathrm{Der}\,\mathcal{D} = \{ d \in \mathrm{Der}\,\mathcal{D} \mid d(\mathcal{D}) \equiv 0 \ (\mathrm{mod}\ \hbar^{p+1}) \}$

(Lie alg. ideal) $\triangleleft\ \mathrm{Der}\,\mathcal{D}$

$$\Longrightarrow\ 0 \to \mathbb{C}[\![\hbar]\!]/\hbar^{p+1} \to \underbrace{G/\hbar^{p+1}G}_{G_p} \to \underbrace{\mathrm{Der}\,\mathcal{D}/(\mathrm{Der}\,\mathcal{D})_{>p}}_{(\mathrm{Der}\,\mathcal{D})_p} \to 0$$

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \hbar^p\mathbb{C} \arrow[r] \arrow[d, "\hbar"'] & \hbar^p A \arrow[r] \arrow[d] & \hbar^p H \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{C}[\hbar]/\hbar^{p+2} \arrow[r] \arrow[d] & G_{p+1} \arrow[r] \arrow[d] & (\mathrm{Der}\,\mathcal{D})_{p+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{C}[\hbar]/\hbar^{p+1} \arrow[r] \arrow[d] & G_p \arrow[r] \arrow[d] & (\mathrm{Der}\,\mathcal{D})_p \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

RHS column is nontrivial, except when $p = 0$:

Lemma 3.6 $0 \to H \to (\mathrm{Der}\,\mathcal{D})_1 \to (\mathrm{Der}\,\mathcal{D})_0 \to 0$ splits into $(\mathrm{Der}\,\mathcal{D})_1 \simeq H \rtimes (\mathrm{Der}\,\mathcal{D})_0$

Pf:

(Same on group level).

§4. $\rightarrow$ central ext^{n} of HC-pairs

$$(X,\Omega) \rightsquigarrow \mathcal{M}_s \in H^1(X, \langle \mathrm{Sympl}, H\rangle).$$

$\forall\ \mathcal{M}_q \in Q(X,\Omega) \simeq H^1(X, \langle \mathrm{Aut}\mathcal{D}, \mathrm{Der}\mathcal{D}\rangle)$

$$\left(0 \to \mathbb{C}[\![\hbar]\!] \to G \to \langle \mathrm{Aut}\mathcal{D}, \mathrm{Der}\mathcal{D}\rangle \to 0 \right)$$

$\rightsquigarrow$ $\langle \mathrm{Aut}\mathcal{D}, \mathrm{Der}\mathcal{D}\rangle$-torsor $\mathcal{M}_q$

$\rightsquigarrow$ obstruction to lifting to G-torsor

$\mathrm{Per}(\mathcal{M}_q) \in H^2_{DR}(X)[\![\hbar]\!]$ (i.e. connecting homomorphism)

called non-commutative period.

Proof of theorem.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \hbar^p\mathbb{C} \arrow[r] \arrow[d] & \hbar^p\cdot\mathcal{A} \arrow[r] \arrow[d] & \hbar^p\cdot H \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{C}[\hbar]/\hbar^{p+2} \arrow[r] \arrow[d] & G_{p+1} \arrow[r] \arrow[d] & (\mathrm{Der}\mathcal{D})_{p+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{C}[\hbar]/\hbar^{p+1} \arrow[r] \arrow[d] & G_p \arrow[r] \arrow[d] & (\mathrm{Der}\mathcal{D})_p \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

- Integrate to HC-pairs levels. (Keep notations)

- Aim: Prove Per 1-1, order-by-order

$$\mathrm{Per}_p : H^1_{\mathcal{M}_s}(X, \langle (\mathrm{Aut}\mathcal{D})_p, (\mathrm{Der}\mathcal{D})_p\rangle) \longrightarrow H^2_{DR}(X)[\![\hbar]\!]/\hbar^p$$

- $(\mathrm{Der}\mathcal{D})_0 = H$

$$\Rightarrow H^1_{\mathcal{M}_s}(X, \langle (\mathrm{Aut}\mathcal{D})_0, (\mathrm{Der}\mathcal{D})_0\rangle) = \{\mathcal{M}_s\}$$

Claim: $\mathrm{Per}_0(Q_0) = [\Omega]$

Pf. of claim:

$$\langle \mathrm{Sympl}\mathcal{A}, H\rangle \to \mathcal{M}_s \to X \longleftrightarrow (X,\Omega)$$

$$\begin{array}{ccc} H^2(\langle \mathrm{Sympl}\mathcal{A}, H\rangle, \mathbb{C}) & \xrightarrow{\mathrm{Loc}} & H^2_{DR}(X,\mathbb{C}) \\ \Uparrow & & \Uparrow \\ [\omega] & \longmapsto & [\Omega] \end{array}$$

$$\omega \text{ from } 0 \to \mathbb{C} \to \mathcal{A} \to H \to 0$$

Induction on p: Given $\mathcal{M} \in H^1_{\mathcal{M}_s}(X, \langle (\mathrm{Aut}\,D)_p, (\mathrm{Der}\,D)_p\rangle)$

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & \hbar^p\mathbb{C} \arrow[r] \arrow[d] & \hbar^p\mathcal{A} \arrow[r] \arrow[d] & \hbar^p\cdot H \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathbb{C}[\hbar]/\hbar^{p+2} \arrow[r] \arrow[d] & G_{p+1} \arrow[r] \arrow[d] & (\mathrm{Der}\,D)_{p+1} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathbb{C}[\hbar]/\hbar^{p+1} \arrow[r] \arrow[d] & G_p \arrow[r] \arrow[d] & (\mathrm{Der}\,D)_p \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

$$\Rightarrow 0 \to \underbrace{(\mathbb{C}[\hbar]/\hbar^{p+2} \oplus \hbar^p\mathcal{A})/\hbar^p\mathbb{C}}_{V} \to G_{p+1} \to (\mathrm{Der}\,D)_p \to 0$$

$$0 \longrightarrow \underbrace{U}_{\mathbb{C}[\hbar]/\hbar^{p+2}} \longrightarrow V \longrightarrow \underbrace{W}_{\mathcal{A}/\mathbb{C} \simeq H} \longrightarrow 0$$

$$\overset{\mathcal{M}}{\rightsquigarrow} \quad 0 \longrightarrow \mathcal{U} \longrightarrow \mathcal{V} \longrightarrow \mathcal{W} \longrightarrow 0 \quad / X$$

$$\rightsquigarrow \cdots \to \underbrace{H^1_{DR}(X,\mathcal{W})}_{H^2_F(X)} \xrightarrow{\delta} H^2_{DR}(X,\mathcal{U}) \to H^2_{DR}(X,\mathcal{V}) \xrightarrow{\beta} H^2_{DR}(X,\mathcal{W}) \to \cdots$$

$$(C' \overset{?}{\longmapsto}) \quad \overset{\Uparrow}{C} \text{ : obstr. of lifting } \mathcal{M} \text{ to } G_{p+1}$$

Lemma 4.2: X admissible $\Rightarrow$ $\beta = 0$, δ : 1-1

$$\left(\exists \overset{\Downarrow}{\text{lift}}\right) \qquad \left(\overset{\Downarrow}{H^2_F(X)\text{-torsor}}\right)$$

Proof of lemma: Want $H^\ell(X,U) \xrightarrow{0} H^\ell(X,W)$ for $\ell = 1,2$.

$$\langle (\mathrm{Aut}\,\mathcal{D})_p, (\mathrm{Der}\,\mathcal{D})_p \rangle \longrightarrow \langle \mathrm{Sympl}\,\mathcal{A}, H \rangle \curvearrowright U, V, W$$

($\Rightarrow$ $\mathcal{U}, \mathcal{V}, \mathcal{W}$ depends only on $\mathcal{M}_s$, not $\mathcal{M}$)

$$\mathbb{C}[\hbar]/\hbar^{p+2} = \mathbb{C}\hbar^{p+1} \oplus \mathbb{C}[\hbar]/\hbar^{p+1}$$

$$\Rightarrow \begin{tikzcd} \mathbb{C} \arrow[d] & & & \\ \mathcal{A} \arrow[d] & V \arrow[d] & = V' \oplus & \mathbb{C}[\hbar]/\hbar^{p+1} \arrow[d] \\ H = & W & & 0 \; ! \end{tikzcd}$$

So enough to show, for $\ell = 1,2$,

$$H^\ell_{DR}(X) \to \underbrace{H^\ell_{DR}(X, \mathrm{Loc}(\mathcal{M}_s, \mathcal{A}))}_{H^\ell(X, \mathcal{O}_X)} \xrightarrow{0?} H^\ell_{DR}(X, \mathrm{Loc}(\mathcal{M}_s, H)) \to H^{\ell+1}_{DR}(X)$$

i.e. onto? $\qquad$ 1-1?

$\Uparrow$

admissibility of X $\qquad$ ▨