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Quantization of line bdl. on Lagr. subvar.

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(X, ω) smooth alg. sympl. variety

Deformation quantization $\mathcal{O}_X \overset{\text{formal deform.}}{\underset{\text{of assoc. alg.}}{\rightsquigarrow}} \mathcal{O}_\hbar \quad (\mathcal{O}_\hbar / \hbar \mathcal{O}_\hbar \simeq \mathcal{O}_X)$

$\mathcal{L} \overset{?}{\rightsquigarrow} \mathcal{L}_\hbar \quad (\mathcal{L}_\hbar / \hbar \mathcal{L}_\hbar \simeq \mathcal{L})$

Coherent sheaf.

(Gabber) $\exists\, \mathcal{L}_\hbar \pmod{\hbar^3} \Rightarrow \operatorname{Supp} \mathcal{L}_\hbar$ coisotropic.

Assume $\mathcal{L}$: $\mathbb{C} \to L \to Y \overset{\text{Lagr}}{\subset} X$

Will constr. $At(\mathcal{O}_\hbar, Y) \in H^2(Y, \Omega_Y^{\geq 1})$

$\left(\sim \; 0 \to \mathcal{O}_Y \to \mathcal{T}or_1^{\mathcal{O}_\hbar}(\mathcal{O}_Y, \mathcal{O}_Y) \to T_Y \to 0 \right)$

$$H^2(Y, \Omega_Y^{\geq 1}) \to \underbrace{H^2(Y, \Omega_Y^{\bullet})}_{H^2_{DR}(Y)} \xleftarrow{i_Y^*} H^2_{DR}(X)[[\hbar]]$$

$At(\mathcal{O}_\hbar, Y) \mapsto At(\mathcal{O}_\hbar, Y)_{DR} = \omega_1(\mathcal{O}_\hbar)|_Y \leftarrow Per(\mathcal{O}_\hbar) = [\omega] + \hbar \omega_1(\mathcal{O}_\hbar) + \hbar^2 \omega_2(\mathcal{O}_\hbar) + \ldots$

Theorem. (i) $\exists\, \mathcal{L}_\hbar$ iff

$$c_1(L) - \tfrac{1}{2} c_1(K_Y) = At(\mathcal{O}_\hbar, Y) \text{ in } H^2(\Omega_Y^{\geq 1})$$

$$\omega_{\geq 2}(\mathcal{O}_\hbar)|_Y = 0$$

(ii) $Q(X, \omega, Y) \triangleq \{\mathcal{L}_\hbar\}/\cong$ is a torsor over $\{$ flat alg. $\mathcal{O}_Y[[\hbar]]^\times$-torsors $/ Y \}/\cong$

If $At(\mathcal{O}_\hbar, Y) = 0$, $\left(\exists\, \mathcal{L}_\hbar \Leftrightarrow L^2 \otimes K_Y^{-1} \text{ flat} + \omega_{\geq 2}(\mathcal{O}_\hbar)|_Y = 0 \right)$

§2 Basic construction. ("Linear" algebra)

Sympl. v.s. (V^{2n}, ω)

$\rightsquigarrow$ Heisenberg Lie alg. $V + \mathbb{C}\hbar$ $\quad [x,y] = \omega(x,y)\hbar$

deg. $\quad$ 1 $\quad$ 2

$\rightsquigarrow$ Univ. Enveloping alg. $D = \mathcal{U}(V + \mathbb{C}\hbar)$

(Remark: $\bigoplus_{i \geq 0} D^i$ vs $\prod_{i \geq 0} D^i$) $\quad \left(\begin{array}{l} \mathcal{U}(\mathfrak{g}) \\ = \otimes \mathfrak{g} / ab - ba - [a,b] \end{array} \right)$

(If $\omega = 0$, then $D = \mathrm{Sym}^{\bullet}(V + \mathbb{C}\hbar)$.

$$\frac{1}{\hbar} D = \underbrace{\frac{1}{\hbar}\mathbb{C}}_{-2} + \underbrace{\frac{1}{\hbar} V}_{-1} + \underbrace{\mathbb{C} + \mathrm{Sym}^2 V}_{0} + \text{deg} > 0 \text{ term}$$

deg; $\frac{1}{\hbar}\mathbb{C} = \frac{1}{\hbar}\mathrm{Sym}^0 V$

$\rightsquigarrow$ graded Lie alg.

$$\frac{1}{\hbar} D = \underbrace{\left(\tfrac{1}{\hbar} D\right)^{-2} + \left(\tfrac{1}{\hbar} D\right)^{-1}}_{\cong V + \mathbb{C}\hbar} + \underbrace{\left(\tfrac{1}{\hbar} D\right)^{0}}_{\mathbb{C} + [(\frac{1}{\hbar}D)^0, (\frac{1}{\hbar}D)^0]} + \cdots\cdots$$

$[(\frac{1}{\hbar}D)^0, (\frac{1}{\hbar}D)^0] = sp(V)$

$$0 \longrightarrow \mathbb{C}[\![\hbar]\!] \longrightarrow \tfrac{1}{\hbar} D \longrightarrow \mathrm{Der}\, D \longrightarrow 0$$

- $\langle G, \mathfrak{h} \rangle$ Harish-Chandra pair

Eg. $\langle \mathrm{Aut}(D), \mathrm{Der}(D) \rangle$ HC-pair

$\quad \mathrm{Aut}(D) \supseteq Sp(V)$ (linear part) $\qquad \mathrm{Lie}\, \mathrm{Aut}(D) = \mathrm{Der}^{\geq 0}(D)$

$(\frac{1}{\hbar} D)^{\geq 1} =: \mathrm{Lie}\, \mathcal{G}^{\geq 1}$ $\leftarrow$ nonlinear part.

(pro)nilp. alg $\qquad$ (pro)unip. gp.

$\uparrow$ center $\qquad \uparrow$

$\hbar\mathbb{C}[\![\hbar]\!] \qquad 1 + \hbar\mathbb{C}[\![\hbar]\!]$

$$(\tfrac{1}{\hbar} D)^{\geq 0} \xleftarrow{\mathrm{Lie}} \mathcal{G} := \mathbb{C}^{\times} \times (Sp(V) \ltimes \mathcal{G}^{\geq 1})$$

$\rightsquigarrow \langle \mathcal{G}, \frac{1}{\hbar} D \rangle$ HC pair

$$1 \to \langle \mathbb{C}[\![\hbar]\!]^{\times}, \mathbb{C}[\![\hbar]\!] \rangle \to \langle \mathcal{G}, \tfrac{1}{\hbar} D \rangle \longrightarrow \langle \mathrm{Aut}(D), \mathrm{Der}(D) \rangle \to 1$$

- (V,ω) w/ std. basis x_i, y_i's

$\rightsquigarrow D$: alg. gen. by $\hbar, x_i, y_i$'s s.t. $[y_j, x_i] = \delta_{ij}\hbar$ (other $[\ ] = 0$)

subalg. $\uparrow$

$\mathbb{C}[\![x_i, \hbar]\!]$ (or $\mathbb{C}[\![y_i, \hbar]\!]$)

$\Downarrow$

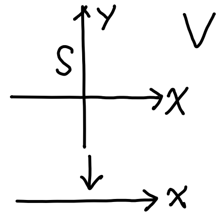
$$y_i f(x) - f(x) y_i = \hbar \frac{\partial}{\partial x_i} f$$

" D ~ a quantization of $A := D/\hbar D \simeq \mathbb{C}[\![x_i, y_i]\!]$ "

A w/ Poisson bracket $\{a,b\} := \frac{1}{\hbar}[a,b]$, $\{f,g\} = \sum_i (\partial_{x_i} f\, \partial_{y_i} g - \partial_{y_i} f\, \partial_{x_i} g)$

- Fix $S \overset{\text{Lagr.}}{\subseteq} V \rightsquigarrow M \triangleq D/DS \curvearrowleft D$

say $\langle y_1, \dots, y_n \rangle$ $\quad \mathbb{C}[\![x_1, \dots, x_n, \hbar]\!]$ $\quad \hbar \frac{\partial}{\partial x_i} = y_i$



- $P \triangleq Sp(V) \cap \{\text{preserve } S\}$ parabolic $P = \begin{pmatrix} * & u \\ 0 & * \end{pmatrix}$ $\begin{matrix} S \\ S^* \end{matrix}$

$\overset{\text{Lie}}{\rightsquigarrow}$ $\mathfrak{p} \leq \mathfrak{sp}(V)$ w/ nilradical $\mathfrak{u}$, $\mathfrak{p}/\mathfrak{u} \simeq \mathfrak{gl}(S)$

Lemma. $a \in \mathfrak{p}$ $\rightsquigarrow \hbar a \in D^2$ $\quad (\hbar a)(1_M) = \frac{1}{2}\operatorname{Tr}(a|_S) \cdot 1_M$ ✩

$(\sim K_Y^{1/2})$

$$D \curvearrowright M = D/D\langle y_1, \dots, y_n \rangle \simeq \mathbb{C}[\![x_1, \dots, x_n, \hbar]\!] \text{ w/ } y_i = \hbar \frac{\partial}{\partial x_i}$$

$$\Rightarrow A = D/\hbar D \curvearrowright M/\hbar M = \mathbb{C}[\![x_1, \dots, x_n]\!] \simeq A/A\langle y_1, \dots, y_n \rangle$$

Conversely,

Lemma 2.3.5. $\forall D \curvearrowright \underline{M}$ finitely generated, w/o $\hbar$-torsion

$$\underline{M}/\hbar\underline{M} \overset{A\text{-mod}}{\simeq} A/AS \Rightarrow \underline{M} \overset{D\text{-mod}}{\simeq} M$$

(i.e. $y = \hbar \frac{\partial}{\partial x}$ is the only way to quantize)

§3. Comparison of HC pairs. (Algebra)

$$S \subseteq V \rightsquigarrow \mathcal{A}S \triangleleft \mathcal{A}$$

$$\rightsquigarrow \mathcal{J} \triangleleft \mathcal{D}$$ (always have y)

require preserves $\mathcal{J}$

$\rightsquigarrow \langle \mathrm{Aut}(\mathcal{D})_{\mathcal{J}}, \mathrm{Der}(\mathcal{D})_{\mathcal{J}} \rangle$ HC-pair

$$1 \to \langle K^\times, K \rangle \to \langle \mathcal{G}_{\mathcal{J}}, (\tfrac{1}{\hbar}\mathcal{D})_{\mathcal{J}} \rangle \longrightarrow \langle \mathrm{Aut}(\mathcal{D})_{\mathcal{J}}, \mathrm{Der}(\mathcal{D})_{\mathcal{J}} \rangle \to 1$$

$K \triangleq \mathbb{C}[\![\hbar]\!]$ $\qquad (\tfrac{1}{\hbar}\mathcal{D})_{\mathcal{J}} = \tfrac{1}{\hbar}\mathcal{J}$ (Lemma)

- $\mathcal{J} = \mathrm{Ann}(M/\hbar M)$ $\quad (\because y f(x) = \hbar \frac{\partial f}{\partial x} \in \hbar\mathbb{C}[x])$

$\Rightarrow \mathcal{J}M \subset \hbar M$

$\Rightarrow \frac{1}{\hbar}\mathcal{J} \curvearrowright M$

$\rightsquigarrow \frac{1}{\hbar}\mathcal{J} \longrightarrow \mathrm{Der}(\mathcal{D}, M) := \{\text{derivat}^{n} \text{ of pair } (\mathcal{D}, M)\}$

Lemma: Lie alg isomorphism

$\rightsquigarrow$ exact seq. of HC-pairs.

$$1 \to \langle K^\times, K \rangle \to \langle \mathrm{Aut}(\mathcal{D}, M), \mathrm{Der}(\mathcal{D}, M) \rangle \longrightarrow \langle \mathrm{Aut}(\mathcal{D})_{\mathcal{J}}, \mathrm{Der}(\mathcal{D})_{\mathcal{J}} \rangle \to 1$$

Prop. 3.2.1.

$$\begin{array}{ccccccccc}
1 \to & \langle \underset{\pm 1}{\underline{K^\times}}, K \rangle & \to & \langle \underset{\pm 1}{\underline{\mathcal{G}_{\mathcal{J}}}}, (\tfrac{1}{\hbar}\mathcal{D})_{\mathcal{J}} \rangle & \longrightarrow & \langle \mathrm{Aut}(\mathcal{D})_{\mathcal{J}}, \mathrm{Der}(\mathcal{D})_{\mathcal{J}} \rangle & \to 1 \\
& \| & & \downarrow \cong & & \| & \\
1 \to & \langle \underset{\pm 1}{\underline{K^\times}}, K \rangle & \to & \langle \underset{\pm 1}{\underline{\mathrm{Aut}(\mathcal{D}, M)}}, \mathrm{Der}(\mathcal{D}, M) \rangle & \longrightarrow & \langle \mathrm{Aut}(\mathcal{D})_{\mathcal{J}}, \mathrm{Der}(\mathcal{D})_{\mathcal{J}} \rangle & \to 1
\end{array}$$

Claim 3.2.2 $\mathcal{G}_g = \mathbb{C}^\times_{\mathcal{G}} \times (\Sigma(P) \ltimes \mathcal{G}_P^{\geq 1})$

Here $\Sigma : Sp(V) \hookrightarrow \mathcal{G}$ canonical embedding

In particular $\mathrm{Lie}\, \mathcal{G}_g = \frac{1}{\hbar} D_g^{\geq 0}$.

'Pf': For unip. part $\mathcal{G}_P^{\geq 1}$, only need to check on Lie alg. level.

Linear part is just P that preserves S (or $\mathfrak{J}$).

Claim 3.2.3 $\mathrm{Aut}(D,M) = \mathcal{E}_{Aut}(\mathbb{C}^\times) \times (\Theta_{D,M}(P) \ltimes \mathrm{Aut}^{\geq 1}(D,M))$

$$\mathcal{E}_{Aut}(\mathbb{C}^\times) \xrightarrow{\ \chi\ } \mathbb{C}^\times, \qquad \Theta_{D,M}(P) \xrightarrow{\text{proj.}} \mathbb{C}^\times \quad (\rightsquigarrow \text{Line bdl. } L)$$

In particular $\mathrm{Lie}\,\mathrm{Aut}(D,M) = \mathrm{Der}^{\geq 0}(D,M)$.

Proof of Prop. 3.2.1.

$$\begin{array}{ccccc}
\mathbb{C}^\times_{\mathcal{G}}/\pm 1 & \times & \Sigma(P) & \ltimes & \mathcal{G}_g^{\geq 1} \\
(1 & , & p & , & g) \\
\downarrow & & & & \\
(\pm\sqrt{\det(p|_S)} & , & p & , & g) \\
\mathcal{E}_{Aut}(\mathbb{C}^\times)/\pm 1 & \times & \Theta_{D,M}(P) & \ltimes & \Phi^{\geq 1}_{D,M}(g)
\end{array}$$

This is why we need $/\pm 1$. Later gives $K_Y^{1/2}$.

The origin of this is: $sp(V) \xrightarrow{\hbar\sigma} D^2$

$$a \in \mathfrak{p} \Longrightarrow (\hbar\sigma(a))1_M = \frac{1}{2}\mathrm{Tr}(a|_S)\cdot 1_M$$

$$\underbrace{\left(\tfrac{1}{\hbar}D\right)^0}_{\mathbb{C} + [(\frac{1}{\hbar}D)^0, (\frac{1}{\hbar}D)^0]} \times \underbrace{\left(\tfrac{1}{\hbar}D\right)^{-1}}_{V} \xrightarrow{[\ \ ]} \left(\tfrac{1}{\hbar}D\right)^{-1}$$

center (act trivially)

$$\Rightarrow \sigma : sp(V) \xrightarrow{\sim} [(\tfrac{1}{\hbar}D)^0, (\tfrac{1}{\hbar}D)^0] \subseteq \tfrac{1}{\hbar}D^2$$

$$\hbar\sigma : sp(V) \longrightarrow D^2$$

Explicitly,

$$\mathfrak{p} \ni a = \begin{pmatrix} g & h \\ 0 & -g^T \end{pmatrix}\begin{matrix} X \\ Y \end{matrix} \quad \text{w/} \quad h = h^T \in \mathfrak{u}, \quad g \in \mathfrak{gl}(S).$$

$$\text{via} \quad sp(V) \simeq Sym^2 V \longrightarrow D^2 \curvearrowright M = D/DS \ni 1_M \qquad (M = \mathbb{C}[[x_\cdot, \hbar]])$$

$$a \longleftrightarrow \underbrace{\frac{1}{2}\sum g_{ij}(x_i y_j + y_j x_i) + \frac{1}{2} h_{ij} y_i y_j}_{\hbar\sigma(a)}$$

Since $x(1) = x$, $y(1) = 0$, $y(x) = \hbar\frac{\partial}{\partial x}(x) = \hbar$

$$\hbar\sigma(a)\cdot 1_M = \frac{1}{2}\sum g_{ii} = \frac{1}{2}Tr(a|_S)\, 1_M \quad (\hbar?)$$

$$\begin{array}{ccc} \mathfrak{p} & \xrightarrow{\ \sigma\ } & \frac{1}{\hbar}\mathfrak{g} \\ & {\scriptstyle(\theta_D, \theta_M)} \searrow & \downarrow {\scriptstyle(\varphi_D, \varphi_M)} \\ & & Der(D, M) \end{array}$$

Commute for D-part, not M-part:

$$(\varphi_M \circ \sigma)(a) = \theta_M(a) + \frac{1}{2}Tr(a|_S)\cdot 1_M$$

§4 HC torsors

Def: $\langle G,\mathfrak{g}\rangle \to P \to Y$ $\quad$ $\langle G,\mathfrak{g}\rangle$-torsor

Def: transitive $\quad$ (see [BK])

$$c: 1 \to \langle \mathfrak{a},\mathfrak{a}\rangle \to \langle \tilde{G},\tilde{\mathfrak{g}}\rangle \to \langle G,\mathfrak{g}\rangle \to 1 \quad \text{w/ v.s. } \mathfrak{a}$$

$$\rightsquigarrow \mathrm{Loc}(\mathcal{P}, c) \in H^2_{dR}(Y) \otimes \mathfrak{a}$$

obstruct^n for lifting $\mathcal{P}$ to a $\langle \tilde{G},\tilde{\mathfrak{g}}\rangle$-torsor$/Y$

- If $c: 1 \to \langle \mathbb{C}^\times, \mathbb{C}\rangle \to \langle \tilde{G},\tilde{\mathfrak{g}}\rangle \to \langle G,\mathfrak{g}\rangle \to 1$

<u>Suppose</u> $\tilde{G} = \mathbb{C}^\times \times G$ (but $\tilde{\mathfrak{g}} \neq \mathbb{C} + \mathfrak{g}$),

Then $\mathrm{Loc}(P,c) \in H^2_{dR}(Y)$ has a lift

to $\alpha(P,c) \in H^2(\Omega^{\geq 1})$

Idea of proof (use Beilinson-Bernstein)

$$0 \to \mathbb{C} \to \tilde{\mathfrak{g}} \to \mathfrak{g} \to 0$$

$$\rightsquigarrow 0 \to \mathcal{O}_P \to \tilde{\mathfrak{g}} \otimes \mathcal{O}_P \to T_P \to 0$$

Lie algebroid $\quad$ anchor map

$\rightsquigarrow$ G-equivar. Picard algebroid $/P$ $\quad \downarrow f$

$$\rightsquigarrow \frac{[f_*(\tilde{\mathfrak{g}} \otimes \mathcal{O}_Z)]^G}{[f_*(\mathrm{Lie}G \otimes \mathcal{O}_Z)]^G} \quad \text{Picard algebroid}/Y$$

$$\xrightarrow[\text{class}]{\text{Atiyah}} \alpha \in H^2(\Omega_Y^{\geq 1})$$

Lemma 4.2.4. As above,

$$\begin{array}{lll} & \langle \tilde{G},\tilde{\mathfrak{g}}\rangle \longrightarrow P \longrightarrow Y & \text{transitive torsor} \\ \xleftrightarrow{\text{up to} \cong} & \left\{ \begin{array}{l} \langle G,\mathfrak{g}\rangle \longrightarrow Z \longrightarrow Y \\ \mathbb{C}^\times \longrightarrow L \longrightarrow Y \end{array} \right. & \begin{array}{l} \text{transitive torsor} \\ \mathbb{C}^\times\text{-torsor} \end{array} \end{array}$$

given by $P \longmapsto (\mathbb{C}^\times \backslash P,\ G \backslash P)$

- $\overline{\mathcal{G}_{\mathfrak{g}}} := \mathcal{G}_{\mathfrak{g}} / \mathbb{C}^{\times}_{\mathcal{G}}$, $\overline{(\frac{1}{\hbar}D)_{\mathfrak{g}}} := (\frac{1}{\hbar}D)_{\mathfrak{g}} / \mathbb{C}_{\mathfrak{g}}$

  $\cong \Sigma(P) \ltimes \mathcal{G}_{\mathfrak{g}}^{\geq 1}$ by Claim 3.2.2

  $\Longrightarrow \mathcal{G}_{\mathfrak{g}} = \mathbb{C}^{\times} \times (\Sigma(P) \ltimes \mathcal{G}_{\mathfrak{g}}^{\geq 1}) = \mathbb{C}^{\times} \times \overline{\mathcal{G}_{\mathfrak{g}}}$ split by $\imath_{\mathfrak{g}}$

- $\langle \overline{\mathrm{Aut}(D,M)}, \overline{\mathrm{Der}(D,M)} \rangle := \langle \frac{\mathrm{Aut}(D,M)}{\mathcal{E}_{\mathrm{Aut}}(\mathbb{C}^{\times})} \; \frac{\mathrm{Der}(D,M)}{\mathcal{E}_{\mathrm{Aut}}(\mathbb{C})} \rangle$

  Claim 3.2.3 $\Longrightarrow$

  $\mathrm{Aut}(D,M) = \mathbb{C}^{\times} \times \overline{\mathrm{Aut}(D,M)}$ split by $\imath_{\mathrm{Der}}$

$$\begin{array}{ccccc}
\rightsquigarrow \langle \frac{K^{\times}}{\mathbb{C}^{\times}}, \frac{K}{\mathbb{C}} \rangle & \hookrightarrow & \langle \overline{\mathcal{G}_{\mathfrak{g}}}, \overline{(\frac{1}{\hbar}D)_{\mathfrak{g}}} \rangle & \twoheadrightarrow & \langle \mathrm{Aut}(D)_{\mathfrak{g}}, \mathrm{Der}(D)_{\mathfrak{g}} \rangle \\
\| & & \cong \downarrow \bar{\Phi}_{D,M}, \bar{\varphi}_{D,M} & & \| \\
\langle \frac{K^{\times}}{\mathbb{C}^{\times}}, \frac{K}{\mathbb{C}} \rangle & \hookrightarrow & \langle \overline{\mathrm{Aut}(D,M)}, \overline{\mathrm{Der}(D,M)} \rangle & \twoheadrightarrow & \langle \mathrm{Aut}(D)_{\mathfrak{g}}, \mathrm{Der}(D)_{\mathfrak{g}} \rangle
\end{array}$$

Prop: $\forall \; \langle \overline{\mathrm{Aut}}(D,M), \overline{\mathrm{Der}}(D,M) \rangle \longrightarrow Z \longrightarrow Y$,

$\Longrightarrow \alpha(Z, c_{\mathrm{Der}}, \imath_{\mathrm{Der}}) - \alpha(\bar{\Phi}_{*}Z, c_{\mathfrak{g}}, \imath_{\mathfrak{g}}) = \frac{1}{2} c_1(L_Z) \in H^2(\Omega_Y^{\geq 1})$

where $\mathbb{C}^{\times} \to L_Z \to Y$ is induced from Z by

$\overline{\mathrm{Aut}}(D,M) \longrightarrow \mathrm{Aut}(D)_{\mathfrak{g}} \longrightarrow \Sigma(P) \xrightarrow{\det} \mathbb{C}^{\times}$

§5. Torsors associated w/ a quantization.

$x \in X$ smooth sympl. variety

(1) formal sympl. coord at x $\Leftrightarrow$ $\hat{\mathcal{O}}_x \xrightarrow[\cong]{\eta} \mathcal{A}$ as Poisson alg.

$\rightsquigarrow \langle \mathrm{Aut}(\mathcal{A}), \mathrm{Der}(\mathcal{A}) \rangle \longrightarrow \mathcal{P}_X \longrightarrow X$

(2) $\mathcal{O}_\hbar$: formal quantizat^n of $\mathcal{O}_X$ $\rightsquigarrow$ $\mathcal{O}_{x,\hbar} \xleftarrow[\cong]{\eta_\hbar} \mathcal{D}$ as $\mathbb{C}[\![\hbar]\!]$-alg

$\rightsquigarrow \langle \mathrm{Aut}(\mathcal{D}), \mathrm{Der}(\mathcal{D}) \rangle \longrightarrow \mathcal{P}_\hbar \longrightarrow X$

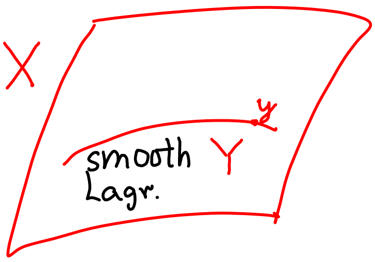
(3) $\mathcal{D} \twoheadrightarrow \mathcal{A} = \mathcal{D}/\hbar\mathcal{D}$ $\rightsquigarrow$ canon. proj.

def. quant. $\mathcal{O}_\hbar \longleftrightarrow$ lift of $\mathcal{P}_X$ to $\mathcal{P}_\hbar$

$$1 \longrightarrow \langle \mathbb{C}[\![\hbar]\!], \mathbb{C}[\![\hbar]\!] \rangle \longrightarrow \langle \mathfrak{g}^+, \tfrac{1}{\hbar}D \rangle \longrightarrow \langle \mathrm{Aut}D, \mathrm{Der}D \rangle \longrightarrow 1$$

(4) Period

$$\mathrm{per}(\mathcal{O}_\hbar) := \mathrm{Loc}(\mathcal{P}_\hbar, \overbrace{\mathfrak{g}^+}^{\mathbb{C}^\times(Sp(V) \ltimes \mathfrak{g}^{\geq 1})}, \tfrac{1}{\hbar}\mathcal{D}) \in H^2_{DR}(X)$$



$$\begin{array}{ccc} X & \hookleftarrow & Y \\ \mathcal{O}_X & \rhd & \mathcal{I}_Y \\ \uparrow & \square & \uparrow \\ \mathcal{O}_\hbar & \longleftarrow & \mathcal{J}_Y \end{array}, \quad \mathcal{O}_{y,\hbar} \overset{\exists \eta}{\simeq} D \supset J \simeq \mathcal{J}_{Y,y}$$

All η's $\rightsquigarrow \langle \mathrm{Aut}(D)_J, \mathrm{Der}(D)_J \rangle \longrightarrow \mathcal{P}_Y \longrightarrow Y$ transitive.

$$1 \longrightarrow \langle \frac{K^\times}{\mathbb{C}^\times}, \frac{K}{\mathbb{C}} \rangle \longrightarrow \langle \overline{\mathfrak{g}_J}, \overline{(\tfrac{1}{\hbar}D)_J} \rangle \longrightarrow \langle \mathrm{Aut}(D)_J, \mathrm{Der}(D)_J \rangle \longrightarrow 1$$

$(K = \mathbb{C}[\![\hbar]\!])$

$\rightsquigarrow$ obstruction class

$$\mathrm{Loc}(\mathcal{P}_J, \overline{\mathfrak{g}_J}, \overline{(\tfrac{1}{\hbar}D)_J}) \in H^2(Y)[\![\hbar]\!]$$

Lemma 5.2.2 $\left(\hbar^2 \omega_2(\mathcal{O}_\hbar) + \hbar^3 \omega_3(\mathcal{O}_\hbar) + \cdots\cdots \right)\Big|_Y$

$\left((\hbar\omega_1)|_Y \text{ disappear } \because / \langle \mathbb{C}, \mathbb{C} \rangle \right)$.

• Given $\mathcal{O}_\hbar$ and $Y \hookrightarrow X$

$$\rightsquigarrow \quad \begin{array}{ccc} \mathcal{O}_\hbar & \twoheadrightarrow & \mathcal{O}_X \\ \uparrow & \square & \uparrow \\ \mathcal{J}'_Y & \longrightarrow & I_Y^2 \end{array}$$

preimage of I_Y^2 in $\mathcal{O}_\hbar$ $\rightarrow \mathcal{J}'_Y$, then $\mathcal{J}_Y^2 \subset \mathcal{J}'_Y \subset \mathcal{J}_Y$.

Lemma 5.3.1. $\underbrace{\underbrace{\mathcal{J}_Y^2 \subset \mathcal{J}'_Y}_{\mathcal{O}_Y} \subset \mathcal{J}_Y}$ with $\mathcal{J}'_Y \subset \mathcal{J}_Y$ giving T_Y, and the whole $\mathcal{J}_Y^2 \subset \mathcal{J}_Y$ giving $\mathrm{Tor}_1^{\mathcal{O}_\hbar}(\mathcal{O}_Y, \mathcal{O}_Y)$.

$$\left(\begin{array}{llll} \mathcal{O}_\hbar & \mathbb{C}[x,y,\hbar] & \mathcal{O}_X & \mathbb{C}[x,y] \\ \mathcal{J}_Y & x\mathbb{C}[x,y]+\hbar\mathbb{C}[x,y,\hbar] & I_Y & x\mathbb{C}[x,y] \\ \mathcal{J}' & x^2\mathbb{C}[x,y]+\hbar\mathbb{C}[x,y,\hbar] & I_Y^2 & x^2\mathbb{C}[x,y] \end{array} \right.$$

$$\frac{\mathcal{J}'}{\mathcal{J}_Y^2} = \frac{\cancel{x^2\mathbb{C}[x,y]} + \hbar\,\mathbb{C}[x,y,\hbar]}{\cancel{x^2\mathbb{C}[x,y]} + x\hbar\,\mathbb{C}[x,y,\hbar] + \hbar^2\mathbb{C}[x,y,\hbar]}$$

$\hbar^{\geq 2}$ terms cancel too

$$= \frac{\hbar\,\mathbb{C}[x,y]}{\hbar x\,\mathbb{C}[x,y]} = \mathbb{C}[y]$$

Lie algebra str. (from $\{I_Y, I_Y\} \subset I_Y$ ($\because$ Lagr.) & $\mathcal{O}_\hbar \times \mathcal{O}_\hbar \xrightarrow{\frac{1}{\hbar}[\,,\,]} \mathcal{O}_\hbar$)

$$0 \to \frac{\mathcal{J}'_Y}{\mathcal{J}_Y^2} \longrightarrow \frac{\mathcal{J}_Y}{\mathcal{J}_Y^2} \longrightarrow \frac{\mathcal{J}_Y}{\mathcal{J}'_Y} \longrightarrow 0$$ Lie alg. exact seq.

$$= \; 0 \to \mathcal{O}_Y \longrightarrow \mathrm{Tor}_1^{\mathcal{O}_\hbar}(\mathcal{O}_Y, \mathcal{O}_Y) \longrightarrow T_Y \to 0$$

$$\left(= \; 0 \to \mathcal{O}_Y \otimes_{\mathcal{O}_\hbar} \mathrm{Tor}_1^{\mathcal{O}_\hbar}(\mathcal{O}_X, \mathcal{O}_X) \to \mathrm{Tor}_1^{\mathcal{O}_\hbar}(\mathcal{O}_Y, \mathcal{O}_Y) \to \mathrm{Tor}_1^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Y) \to 0 \right)$$

$\rightsquigarrow$ Atiyah class $\mathrm{At}(\mathcal{O}_\hbar, Y) \in H^2(\Omega_Y^{\geq 1})$.

Lemma 5.3.5 (i) $\mathrm{At}(\mathcal{O}_\hbar, Y) = \alpha(P_{\mathcal{J}}, \tilde{c}, \tilde{\imath})$

(ii) In $H^2_{DR}(Y)$, $\mathrm{At}(\mathcal{O}_\hbar, Y)_{DR} = \omega_1(\mathcal{O}_\hbar)|_Y$

§ 6. Proof of the main result

Given $Y \overset{\text{Lagr.}}{\hookrightarrow} X$

Lemma 6.1.1. L w/ $L_\hbar$

$\longleftrightarrow$ lift of $\langle \mathrm{Aut}(D)_g, \mathrm{Der}(D)_g \rangle \to P_g \longrightarrow Y$

to $\langle \mathrm{Aut}(D,M), \mathrm{Der}(D,M) \rangle \to P_{D,M} \longrightarrow Y$ transitive

$(\rightsquigarrow L \simeq P_{D,M} \otimes_\chi \mathbb{C}^\times$ via $\chi : \mathrm{Aut}(D,M) \longrightarrow \mathbb{C}^\times)$

Intermediate lift is

$\langle \overline{\mathrm{Aut}}(D,M), \overline{\mathrm{Der}}(D,M) \rangle \longrightarrow \overline{P}_{D,M} \longrightarrow Y$ transitive

Lemma 6.1.4 $\exists\ \overline{P}_{D,M}$

$\Longleftrightarrow (\hbar^2 \omega_2(\mathcal{O}_\hbar) + \hbar^3 \omega_3(\mathcal{O}_\hbar) + \dots)|_Y = 0 \in H^2(Y)[\![\hbar]\!]$

Lemma 6.2.1 $\exists\ P_{D,M}$ lifting $\overline{P}_{D,M}$

$\Longleftrightarrow c_1(L) - \frac{1}{2} c_1(K_Y) = At(\mathcal{O}_\hbar, Y)$

$(P_{D,M} = L \times_Y \overline{P}_{D,M} \quad \exists L)$