Costello. SUSY field theories for mathematicians

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		fermion	R-Symmetry
$n=2$	$S^2 S_+ \longrightarrow V$ $S^2 S_- \longrightarrow V$	(n,m) SUSY $S_+\otimes\mathbb{C}^n + S_-\otimes\mathbb{C}^m$	$SO(n)\times SO(m)$
$n=3$	$S^2 S \longrightarrow V$	n SUSY $S\otimes\mathbb{C}^n$	$SO(n)$
$n=4$	$S_+\otimes S_- \rightarrow V$	n SUSY $S_+\otimes\mathbb{C}^n + S_-\otimes\mathbb{C}^{n*}$	$GL(n)$

$n=5$

$Spin(5,\mathbb{C}) = Sp(4,\mathbb{C})$

$S=\mathbb{C}^4$ = fund. rep. of $Sp(4,\mathbb{C})$.

$V = (\Lambda^2 S)/\mathbb{C}\omega \subset \Lambda^2 S$, ω=sympl. form.

$S\otimes\mathbb{C}^{2n}$, n extended SUSY, $\mathbb{C}^{2n}$ is symplectic

R-symmetry is $Sp(2n,\mathbb{C})$.

$n=6$ $Spin(6,\mathbb{C}) = SL(4,\mathbb{C})$

$S=\mathbb{C}^4$ = fund. repn. of $SL(4,\mathbb{C})$.

$V=\Lambda^2 S_+$ (n,m) extended SUSY

$S_- = S_+^*$ $S_+\otimes\mathbb{C}^{2n} + S_-\otimes\mathbb{C}^{2m}$

$V=\Lambda^2 S_-$ R-symmetry $Sp(2n,\mathbb{C})\times Sp(2m,\mathbb{C})$

$(1,0)$ SUSY $S_+\otimes\mathbb{C}^2$ 8 supercharges

Example: M5 brane in 11d has $(2,0)$ SUSY,

Rotation of normal directions in R-symmetry

Expect $Spin(5,\mathbb{C})$ R-symmetry.

We found $Sp(4,\mathbb{C}) = Spin(5,\mathbb{C})$.

- Low dimensional SUSY

  d even S_+, S_- of dim. $2^{d/2-1}$

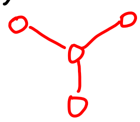
  d odd S of dim $2^{(d-1)/2}$

  V vector repn.

$d=7$. S, $V \subset \Lambda^2 S$, $\dim S = 8$

n SUSY $S \otimes \mathbb{C}^{2n}$, R-symmetry is $Sp(2n,\mathbb{C})$

$d=8$. S_+, S_- 8 dimensional



$V \subseteq S_+ \otimes S_-$

n extended SUSY $S_+ \otimes \mathbb{C}^n + S_- \otimes \mathbb{C}^{n*}$

R-symmetry is $GL(n,\mathbb{C})$.

$d=10$ S_+, S_- dim 16. $V \subset S^2 S_+$, $V \subset S^2 S_-$.

(n,m) SUSY $S_+ \otimes \mathbb{C}^n + S_- \otimes \mathbb{C}^m$

R-symmetry is $SO(n) \times SO(m)$.

$(1,0)$ 10d super YM

$(2,0)$ IIB string

$(1,1)$ IIA string

In dim $d = 4, 6, 10$, there exists a SUSY gauge theory w/ minimal SUSY $N=1$ 4d $(1,0)$ 6d $(1,0)$ 10d

Only fields are a connection $A \in \Omega^1(M^d, \mathfrak{g})$

$\psi \in C^\infty(M, S \otimes \mathfrak{g})$. section of spin bundle

$d=4$ $S = S_+ \oplus S_-$; $d=6$ $S = S_+$; $d=10$ $S = S_+$

Action is $\int F(A) \wedge * F(A) + \int \langle \psi, \not{D} \psi \rangle$

$\not{D} : C^\infty(M, S_\pm) \longrightarrow C^\infty(M, S_\mp)$ is the composition

$$C^\infty(M, S_\pm) \xrightarrow{\nabla} C^\infty(M, T^*_M \otimes S_\pm) \xrightarrow{\mathcal{C}\ell} C^\infty(M, S_\mp)$$

Clifford multi.

Action of SUSY alg.

Work in flat space. $Q \in S$ defines a symmetry of space of fields by

$$(A, \psi) \longrightarrow (A + \varepsilon \Gamma(Q,\psi), \psi + \varepsilon F(A) \cdot Q)$$

Clifford multi (=rotation).

$\Gamma(Q,\psi) \in C^\infty(\mathbb{R}^d, T_{\mathbb{R}^d} \otimes \mathfrak{g})$ $\simeq \Omega^1(\mathbb{R}^d) \otimes \mathfrak{g}$

$F(A) \in \Omega^2(\mathbb{R}^d) \otimes \mathfrak{g}$

$\Lambda^2 \mathbb{R}^d \subseteq C\ell(\mathbb{R}^d)$ a copy of $\mathfrak{so}(d,\mathbb{C})$.

Claim. In dim. $d=4,6,10$

1) This infinitesimal symmetry preserves the action function

2) If $\mathcal{V}_Q$ = vector field on space of fields associated to ,

then $[\mathcal{V}_Q, \mathcal{V}_{Q'}] = \mathcal{L}_{\Gamma(Q\otimes Q')}$ modulo gauge symmetry + on the sol^n. EOM

Lie derivative

M manifold

$C^\infty(M,S) \supseteq$ cov. constant spinors
$C^\infty(M,TM) \supseteq$ cov. constant vectors
$\Big\}$ these will form a smaller SUSY alg.

$d=4$. Check SUSY commutation relations. $\mathfrak{g}=\mathbb{R}$ abelian.

Fields is now a linear superspace, SUSY acts by linear

$$\Gamma(Q_-,\psi_+)+\Gamma(Q_+,\psi_-) \longleftrightarrow (\psi_+,\psi_-)$$

$$\Omega^1(\mathbb{R}^4) \qquad C^\infty(M, S_+\oplus S_-)$$

$$A \longmapsto (dA\cdot Q_+, dA\cdot Q_-)$$

$$Q=(Q_+,Q_-)\in S_+\oplus S_-$$

Compute $[Q_+,Q_-]$ acts on A

$$Q_-Q_+ : A \longmapsto dA\cdot Q_+ \longmapsto \Gamma(Q_-\otimes(dA\cdot Q_+))$$

$$Q_+Q_- : A \longmapsto dA\cdot Q_- \longmapsto \Gamma(Q_+\otimes(dA\cdot Q_-))$$

Note, $dA(\Gamma(Q_+\otimes Q_-)) = \Gamma((dAQ_+)\otimes Q_-) + \Gamma(Q_+\otimes dA\cdot Q_-)$

$$[Q_+,Q_-] = dA\cdot\Gamma(Q_+\otimes Q_-) = \Gamma(Q_+\otimes Q_-)\vee dA$$

$$= \mathcal{L}_{\Gamma(Q_+\otimes Q_-)}A + d(\Gamma(Q_+\otimes Q_-)\vee A)$$

This is a gauge transformation

Eg. What is field content of $N=4$ SYM in $d=4$?
What are linearized SUSY transformations?

Ans: $N=4, d=4$ is reduced from $N=(1,0), d=10$

Field content $A_{4d}\in\Omega^1(\mathbb{R}^4)$ $\varphi_1,\dots,\varphi_6\in C^\infty(\mathbb{R}^4)$

$$A_{10d} = A_{4d} + \sum_{i=1}^{6} dx_{4+i}\cdot\varphi_i$$

10d, 16 spinors in S_+^{10d}, S_+^{10d} decomposes under action of

$Spin(4)\times Spin(6)$ as $S_+^{10d} = S_+^{4d}\otimes S_+^{6d} + S_-^{4d}\otimes S_-^{6d} = S_+^{4d}\otimes W + S_-^{4d}\otimes W^*$

W = fund. rep. of $SL(4,\mathbb{C}) = Spin(6,\mathbb{C})$.

Exercise: Computed linearized SUSY transf.