0-cycles on projective K3 surfaces

Note by Conan 1809.

references:

O'Grady. Moduli of sheaves and the Chow group of K3 surfaces.

Voisin. Rational equivalence of 0-cycles on K3 surfaces and conjectures of Huybrechts and O'Grady.

Shen-Yin-Zhao. Derived categories of K3 surfaces, O'Grady filtration, and 0-cycles in Holomorphic Symplectic varieties.

§ Main theorem.

X proj. K3

$CH_1(X) \simeq H^2(X, \mathbb{Z})$

$CH_0(X)$ very 'large' (Mumford)

- (Beauville-Voisin) $\exists$ canon. $[o_X] \in CH_0(X)$ of deg 1 s.t. $p \in \mathbb{P}^1 \hookrightarrow X \Rightarrow [o_X] = [p]$

- O'Grady filtration $S_0 \subset S_1 \subset \cdots \subset CH_0$ s.t. $S_0 = \mathbb{Z}[o_X]$
  $S_i \triangleq \langle \sum_{j=1}^{i} p_j + \mathbb{Z}[o_X] \rangle$ w/ $p_i \in X$

Main Theorem $(*)$:

$$\forall \mathcal{E} \in D^b(X) \Rightarrow c_2(\mathcal{E}) \in S_{d(\mathcal{E})}(X)$$

$d(\mathcal{E}) = \frac{1}{2} \dim \mathrm{Ext}^1(\mathcal{E}, \mathcal{E})$.

(eg. $\mathcal{E} = \mathcal{O}_{p_1+p_2} \Rightarrow c_2(\mathcal{E}) = p_1 + p_2 \in S_2(X)$).

- Lift $S_\bullet$ to $\tilde{S}_\bullet$ on $CH^\bullet(X) \xrightarrow[\text{proj.}]{} CH^2(X)$

Cor. $\tilde{S}_\bullet$ is preserved under derived equiv.

§ Some lemmas by O'Grady.

X projective K3

Lemma : curve $C \xrightarrow{f} X$ non-const.

$$\Rightarrow f_* CH_0(C) \subset S_{g(C)} X$$

Pf : $\exists\, p \in X$ w/ $f_*[p] = c_X$

$\because \exists$ ample rational $D \subset X$

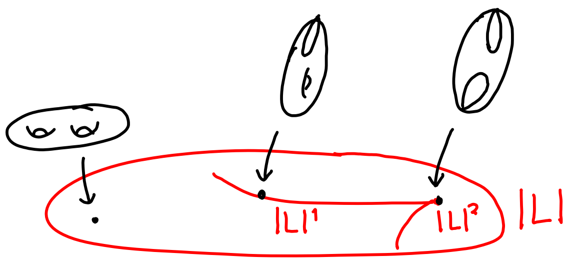
choose $p \in C \cap D$

Recall : $S^g C \twoheadrightarrow Pic^g(C)$ ($\because$ R.R.)

$z \in CH_0(C)$

$$\Longrightarrow z - (|z| - g)p \in Pic^g(C)$$

$\equiv p_1 + \dots + p_g \qquad \exists\, p_i \in C$



Cor : L ample $/ X$

$|L| \supset |L|^\delta \underset{\text{generic}}{\ni} C$, # nodes of $C \geq \delta$

$$\Longrightarrow \iota_* CH_0(C) \subset S_{\underbrace{g(L)-\delta}_{g_0}}(X)$$

Also true for non-generic $C \in |L|^\delta$

($\because$ limit of rat. curves is rat. curve)

Lemma 1: $S_g(X) \xrightarrow{m\times} S_g(X)$

Pf: $Z = p_1 + \dots + p_{g_0} \in X^{(g_0)} \overset{?}{\Longrightarrow} m[Z] \in S_{g_0}(X)$

Take generic (X, L) w/ $g(L) > g_0$

$\Rightarrow \exists\, |L| \ni C \supset p_1 + \dots + p_{g_0}$ (# of conditions is $g_0 + \delta = g(L)$)

and w/ $\delta = g(L) - g_0$ # of nodes.

$\overset{\text{lemma}}{\Longrightarrow}$ $m[Z] \in CH_0(C) \longrightarrow S_{g_0}(X)$ $\square$

Given any distinguished triangle in $D^b(X)$

$$\mathcal{F} \longrightarrow \mathcal{E} \longrightarrow \mathcal{G} \longrightarrow \mathcal{F}[1]$$

$$c_2(\mathcal{E}) = c_2(\mathcal{F} \oplus \mathcal{G})$$

$$\Rightarrow c_2(\mathcal{E}) - c_2(\mathcal{F}) - c_2(\mathcal{G}) = \sum c_1(-) c_1(-) \in S_0$$

$\underset{(m=-1)}{\overset{\text{lemma1}}{\Longrightarrow}}$ (2 of $c_2(\mathcal{F}), c_2(\mathcal{E}), c_2(\mathcal{G})$ in S_i & $S_j \Longrightarrow 3^{rd}$ in S_{i+j})

Next lemma is important for inductive arguments.

Lemma 2: Assume $\mathrm{Hom}(\mathcal{F}, \mathcal{G}) = 0$

$c_2(\mathcal{F}) \in S_{d(\mathcal{F})}$ and $c_2(\mathcal{G}) \in S_{d(\mathcal{G})}$

$\Longrightarrow c_2(\mathcal{E}) \in S_{d(\mathcal{E})}$.

Pf: Recall Mukai: $d(\mathcal{F}) + d(\mathcal{G}) \leq d(\mathcal{E})$

reason: $\mathcal{E} \sim \mathcal{F} \overset{\text{twist.}}{\oplus} \mathcal{G}$

$d(\mathcal{E}) = \dim \mathrm{Ext}^1(\mathcal{E}, \mathcal{E}) = 2 \cdot \dim \mathrm{End}(\mathcal{E}) - \chi(\mathcal{E}, \mathcal{E})$ same for $\mathcal{F} \oplus \mathcal{G}$

$\mathrm{End}(\mathcal{F} \oplus \mathcal{G}) = \mathrm{End}(\mathcal{F}) + \mathrm{End}(\mathcal{G}) + 2\, \underbrace{\mathrm{Hom}(\mathcal{F}, \mathcal{G})}_{0}$

(X, H) polarized K3, $\mathcal{E} \in D^b(X)$

- Recall: $\mathcal{E}$ simple VB $\overset{\text{Voisin}}{\Longrightarrow}$ $(*)$ ✓
- Recall: $\mathcal{E}$ spherical $\overset{\text{Huybrechts}}{\Longrightarrow}$ $(*)$ ✓
  (proof skipped).
- $\mathcal{E}$ torsion free, μ-stable $\Longrightarrow$ $(*)$ ✓

Pf: Recall:

torsion free $\Rightarrow$ codim ≥ 2 singularity

reflexive $\Rightarrow$ codim ≥ 3 singularity

(i.e. VB / surface).

$$0 \longrightarrow \mathcal{E} \longrightarrow \underset{\text{VB}}{\mathcal{E}^{**}} \longrightarrow \underset{\text{0-dim.}}{Q} \longrightarrow 0 \qquad \ell := \text{length}$$

$\mathcal{E}^{**}$ μ-stable ($\because$ stability $\sim c_1 \sim$ codim 1 matter only)

$\Downarrow$

$\Longrightarrow \mathcal{E}^{**}$ simple VB $\overset{\text{Voisin}}{\Longrightarrow}$ $c_2(\mathcal{E}^{**}) \in S_{d(\mathcal{E}^{**})}$

$\ell \leq d(Q)$ ($\because$ each pt. move in 2 dim., + possible ext^{n})

$\Rightarrow C_2(Q) \in S_\ell \subset S_{d(Q)}$

$$Q[-1] \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}^{**} \longrightarrow Q \quad \text{disting. } \triangle$$

$$\mathrm{Hom}(Q[-1], \mathcal{E}^{**}) = \mathrm{Ext}^1(Q, \mathcal{E}^{**}) = 0$$

($\because$ Q 0-dim, $\mathcal{E}^{**}$ VB)

$\overset{\text{lemma2}}{\Longrightarrow}$ $(*)$ ✓

- $\mathcal{F}$ torsion-free, μ-stable

  $\mathcal{E}$: iterated extension of $\mathcal{F}$ $\Longrightarrow$ $(*)$ ✓

Pf:

$$c_2(\mathcal{E}) = c_2(\mathcal{F}^{\oplus m})$$
$$= m\, c_2(\mathcal{F}) + D_1 \cdot D_2$$

$\swarrow$ spanned by $\cap$ divisors $\Rightarrow D_1 \cdot D_2 \in S_0$

$\mathcal{F}$: μ-stable $\Longrightarrow$ $v(\mathcal{F})^2 \geqslant -2$

(i) $v(\mathcal{F})^2 = -2$, i.e. spherical

$$\xrightarrow{\text{Huybrechts}} \quad c_2(\mathcal{F}) \in S_0(X)$$

$$\xrightarrow{\text{Lemma 1.}} \quad c_2(\mathcal{E}) = m\, c_2(\mathcal{F}) + D_1 \cdot D_2 \in S_0(X)$$

(ii) $v(\mathcal{F})^2 \geqslant 0$

$$2\, d(\mathcal{E}) = \underbrace{v(\mathcal{E})^2}_{\geqslant m^2 v(\mathcal{F})^2} + 2\, \underbrace{\dim \mathrm{Hom}(\mathcal{E}, \mathcal{E})}_{\geqslant 1}$$

$$\geqslant v(\mathcal{F})^2 + 2 \overset{\mathcal{F}\text{ stable}}{=} 2\, d(\mathcal{F})$$

$$\Rightarrow \quad c_2(\mathcal{E}) = m\, c_2(\mathcal{F}) + D \underset{\text{\& Prop 1.7}}{\overset{\text{lemma 1.1}}{\in}} S_{d(\mathcal{F})} \subset S_{d(\mathcal{E})} \ ✓$$

$$\Longrightarrow \quad c_2(\mathcal{E}) = m\, c_2(\mathcal{F}) + D_1 \cdot D_2$$

$$\underset{\text{lemma 1.}}{\in} S_{d(\mathcal{F})} \subset S_{d(\mathcal{E})}$$

$\Longrightarrow$ $(*)$ ✓

Prop : $\mathcal{E}$ μ-s.s. VB

$\Longrightarrow \exists \quad 0 \to \mathcal{M} \to \mathcal{E} \to \mathcal{G} \to 0$

$\mathcal{M}$: iterated ext^{n} of μ-stable VB $\mathcal{F} \neq 0$

$\mathcal{G}$: torsion free μ-stable

$\mathrm{Hom}(\mathcal{M}, \mathcal{G}) = 0.$

Pf: WLOG $\mathcal{E}$ semi-stable, Not stable

$\Longrightarrow \exists \underset{\mu\text{-stable}}{\mathcal{F}} \leqslant \mathcal{E}$, $\mu(\mathcal{F}_0) = \mu(\mathcal{E})$

$\mathcal{E}$ VB $\Longrightarrow$ can assume $\mathcal{F}$ VB (take double dual).

$\mathcal{G}_0$: s.s.

$$\begin{tikzcd} & & & \mathcal{T} \arrow[d, hook] & \text{Torsion} \Rightarrow \text{0-dim.} \\ \rightsquigarrow \quad 0 \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{E} \arrow[r] & \mathcal{G}_0 \arrow[r] \arrow[d, two heads] & 0 \quad \leftarrow \mu\text{-s.s.} \\ & & & \mathcal{G}_0^F & \text{Torsion-free} \end{tikzcd}$$

$\Longrightarrow$

$$\begin{tikzcd} & 0 \arrow[d] & & 0 \arrow[d] & \\ & & & \mathcal{T} \arrow[d] & \\ 0 \arrow[r] & \mathcal{F}_{VB} \arrow[r] \arrow[d] & \mathcal{E} \arrow[r] \arrow[d, equal] & \mathcal{G}_0 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{F}' \arrow[r] \arrow[d] & \mathcal{E} \arrow[r] & \mathcal{G}_0^F \arrow[r] \arrow[d] & 0 \\ & \mathcal{T} \arrow[d] & & 0 & \\ & 0 & & & \end{tikzcd}$$

$\mathcal{T}$ torsion $\Longrightarrow (\times) \quad \Longrightarrow \mathcal{T} = 0$

$\Longrightarrow \mathcal{G}_0$ torsion-free

If $\mathrm{Hom}(\mathcal{F}, \mathcal{G}_0) = 0 \Longrightarrow$ ✓

$\because \mathcal{F}$: stable, same slope

If $\mathrm{Hom}(\mathcal{F}, \mathcal{G}_0) \neq 0 \Longrightarrow 0 \to \mathcal{F} \to \mathcal{G}_0 \to \mathcal{G}_1 \to 0$

Similar as above $\rightsquigarrow 0 \to \mathcal{F}_1 \to \mathcal{E} \to \mathcal{G}_1 \to 0$ w/ $0 \to \mathcal{F} \to \mathcal{F}_1 \to \mathcal{F} \to 0$

Inductively $\Longrightarrow$ ✓

- $\mathcal{E}$ torsion free shf $\Rightarrow$ $(*)$ ✓

Pf: Induction on rank$(\mathcal{E})$

$rk(\mathcal{E})=1 \Rightarrow$ μ-stable automatically $\Longrightarrow$ ✓

$rk(\mathcal{E}) \geq 2$.

(Harder-Narasimhan filtration)

- $\mathcal{E}$ not μ-s.s. $\Rightarrow \exists\ 0 \to \underset{0\neq}{\mathcal{F}} \to \mathcal{E} \to \underset{\neq 0}{\mathcal{G}} \to 0$

$\mu(\mathcal{F}) > \mu(\mathcal{E}) > \mu(\mathcal{G})$

($\forall$ μ-stable factor too)

$\Rightarrow \mathrm{Hom}(\mathcal{F}, \mathcal{G}) = 0$

$\xrightarrow[\text{Prop1.4}]{\text{Induc}^{\text{n}}\text{ hypo.}}$ ✓

- $\mathcal{E}$ μ-s.s. (WLOG $\mathcal{E} = \mathcal{E}^{**}$, i.e. VB)

$\xRightarrow{\text{Prop}} \exists\ 0 \to \mathcal{M} \to \mathcal{E} \to \mathcal{G} \to 0$

$\mathcal{M}$: iterated ext^{n} of μ-stable VB $\mathcal{F} \neq 0$

$\mathcal{G}$: torsion free μ-stable

$\mathrm{Hom}(\mathcal{M}, \mathcal{G}) = 0.$

Recall $(*)$ ✓ for $\mathcal{M}$

If $\mathcal{G} = 0 \Longrightarrow \mathcal{E} = \mathcal{M}$ $(*)$ ✓

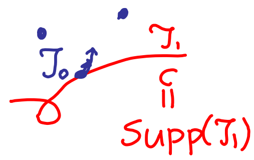
If $\mathcal{G} \neq 0 \xRightarrow{\text{Induction}}$ $(*)$ ✓ for $\mathcal{G}$

$\xRightarrow[\mathrm{Hom}(\mathcal{M},\mathcal{G})=0]{\text{lemma 2}}$ $(*)$ ✓ for $\mathcal{E}$

- $\mathcal{T}$ torsion sheaf $\Longrightarrow$ $(*)$ ✓

Pf:

$$0 \to \underset{\text{0-dim}}{\mathcal{T}_0} \to \mathcal{T} \to \underset{\text{pure 1dim}}{\mathcal{T}_1} \to 0$$



$$\mathrm{Hom}(\mathcal{T}_0, \mathcal{T}_1) = 0$$

$$c_2(\mathcal{T}_0) \in S_{d(\mathcal{T}_0)}(X) \quad ✓$$

$$c_2(\mathcal{T}_1) \in S_{d(\mathcal{T}_1)}(X)$$

reason: $d(\mathcal{T}_1) = \frac{1}{2}\underbrace{v(\mathcal{T}_1)^2}_{\substack{(0,\ell,s)^2 \\ \geq \ell^2}} + \underbrace{\dim \mathrm{Hom}(\mathcal{T}_1, \mathcal{T}_1)}_{\geq 1} \geq \underbrace{\frac{\ell^2}{2} + 1}_{g(C)}$

lemma $\Rightarrow$ $c_2(\mathcal{T}_1) \in S_{g(C)}(X) \subset S_{d(\mathcal{T}_1)}(X)$

$\xRightarrow{\text{lemma 1.}}$ $(*)$ ✓

- $\mathcal{E}$ coherent shf $\Longrightarrow$ $(*)$ ✓

Pf:

$$0 \to \underset{\text{torsion}}{\mathcal{T}} \to \mathcal{E} \to \underset{\text{torsion free}}{\mathcal{F}} \to 0$$

$$\mathrm{Hom}(\mathcal{T}, \mathcal{F}) = 0 \quad ✓$$

Prop 1.4 $\Rightarrow$ ✓

- Any $\mathcal{E} \Longrightarrow (*)$ ✓

Pf: bounded cpx. $\mathcal{E} \in D^b(X)$

$\mathcal{E}$ coherent (shifted) sheaf $\Longleftrightarrow \exists!\, i$ s.t. $h^i(\mathcal{E}) \neq 0$

$\Longleftrightarrow 0 = \ell(\mathcal{E}) \triangleq \max\{|i-j| : h^i(\mathcal{E}) \neq 0 \neq h^j(\mathcal{E})\}$

Induction on $\ell(\mathcal{E})$: $\ell(\mathcal{E}) = 0$ i.e. shf., ✓.

Say $h^m(\mathcal{E}) \neq 0$ + $h^{>m}(\mathcal{E}) = 0$ (say $m=0$)

$\rightsquigarrow \underbrace{\mathcal{F}}_{\tau^{\leq(-1)}\mathcal{E}} \to \mathcal{E} \to \underbrace{\mathcal{G}}_{h^0(\mathcal{E})} \to \mathcal{F}[1]$

$h^0(\mathcal{E}) \rightsquigarrow$ single shf. $\xRightarrow{\text{before}} \mathcal{G}$ $(*)$ ✓

$\ell(\mathcal{F}) < \ell(\mathcal{E}) \xRightarrow{\text{induct}^{\text{n}}} \mathcal{F}$ $(*)$ ✓

deg reason $\Rightarrow \mathrm{Hom}(\mathcal{F}, \mathcal{G}) = 0 \xRightarrow{\text{lemma 1.}} \mathcal{E}$ $(*)$ ✓