1. Construct an immersion $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ that is not a diffeomorphism onto its image.

2. Is $\{(x, y, z) \in \mathbb{R}^3 : x^3 + y^3 + z^3 - 3xyz = 1\}$ a smooth submanifold of $\mathbb{R}^3$?

3. For $0 < r < 1$, compute the area of $D_r = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < r^2\}$ with the metric

$$ds^2 = \frac{1 + x^2}{(1 - x^2 - y^2)^2}\, dx^2 + \frac{1}{(1 + x^2)(1 - x^2 - y^2)^2}\, dy^2.$$

4. Given a Riemannian manifold M, show that there exists a unique affine connection ∇ on M satisfying

$$\nabla_X Y - \nabla_Y X = [X, Y] \text{ and } X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle,$$

for smooth vector fields X, Y, Z on M.

5. The Poincaré half-plane is a hyperbolic plane $\{(x, y) \in \mathbb{R}^2 : y > 0\}$ with metric $g = \frac{1}{y^2}(dx^2 + dy^2)$. It is known that any vertical half-line starting from the x-axis $\sigma(t) = \{(c, f(t)) : c \text{ constant}, f(t) > 0\}$ is a geodesic. Find $f(t)$.

(*Hints*: The equation of a geodesic: $\frac{d^2 x_k}{dt^2} + \sum_{i,j} \Gamma^k_{ij} \frac{dx_i}{dt}\frac{dx_j}{dt} = 0$ $k = 1, 2$;
$\Gamma^k_{ij} = \frac{1}{2}\sum_m \left\{ \frac{\partial}{\partial x_i} g_{jm} + \frac{\partial}{\partial x_j} g_{mi} - \frac{\partial}{\partial x_m} g_{ij} \right\} g^{mk}$;
Among Γ^k_{ij}, you only need Γ^2_{22}.)

6. Let g be a metric on $M = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ and h a metric on $N = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ given by

$$g_{ij} = \frac{4\delta_{ij}}{(1 - x^2 - y^2)^2}, \quad h_{ij} = \frac{\delta_{ij}}{y^2}.$$

Define a diffeomorphism $\phi : M \to N$ by

$$\phi(x, y) = \left(\frac{2x}{x^2 + (1 - y)^2}, \frac{1 - x^2 - y^2}{x^2 + (1 - y)^2} \right).$$

Suppose we know that

$$h\left(d\phi\left(\frac{\partial}{\partial x}\right), d\phi\left(\frac{\partial}{\partial y}\right)\right) = g\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$$

and
$$h\left(d\phi\left(\frac{\partial}{\partial y}\right), d\phi\left(\frac{\partial}{\partial y}\right)\right) = g\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right).$$
Show that ϕ is an isometry.

7. For the Riemmannian curvature
$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$ and
$R(X,Y,Z,W) = \langle R(X,Y)Z, W\rangle$,
show that

   (a) $R(X,Y)Z + R(Z,X)Y + R(Y,Z)X = 0$;
   (Hint: Jacobi identy $[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$)

   (b) $R(X,Y,Z,W) = R(Z,W,X,Y)$.