1. Let v, w be orthonormal vectors in T_pM of a Riemannian manifold M. Then v and w induce parallel vector fields V and W on T_pM. Define

$$\begin{aligned} \rho_s(t) &= (V+sW)t, \;\; a \le t \le b, -\varepsilon < s < \varepsilon \\ \alpha(t,s) &= \exp_p \circ \rho_s(t) : [a,b] \times (-\varepsilon, \varepsilon) \to M, \end{aligned}$$

and let $T = d\alpha(\frac{\partial}{\partial t}), J = d\alpha(\frac{\partial}{\partial s})$. For each s, ρ_s is a ray in T_pM and $\alpha(t,s)$ is a geodesic through p with initial tangent vector $v + sw$.

   (a) Show that the Jacobi field J satisfies the equation

$$\nabla_T \nabla_T J = R(T,J)T.$$

   (b) Let $\gamma(t)$ be a geodesic with unit normal vector field $N(t)$ along γ on $\mathbb{R}^2$. Show that the Jacobi field perpendicular to $\gamma(t)$ satisfies $J(t) = (at+b)N(t)$ for some constants a, b. Find the Jacobi field $J(t)$ perpendicular to a geodesic on $\mathbb{S}^2$ and on $\mathbb{H}^2$.

2. Let $\gamma : [a,b] \to M$ be a geodesic and let $\alpha : Q \to M$ be a C^∞ map with $Q = [a,b] \times [-\varepsilon, \varepsilon] \times [-\delta, \delta]$ and $\alpha(t,0,0) = \gamma(t)$. Define $L(v,w) = \int_a^b ||T|| ds$ to be the arc length of the curve $t \mapsto \alpha(t,v,w)$. Assume $||\gamma'|| \equiv 1$ and denote $T = d\alpha(\frac{\partial}{\partial t}), V = d\alpha(\frac{\partial}{\partial v}), W = d\alpha(\frac{\partial}{\partial w})$. From the first variation formula we know that

$$\frac{\partial}{\partial v} L(v,w) = \int_a^b \frac{\langle \nabla_T V, T \rangle}{||T||}.$$

Derive the second variation formula:

$$\frac{\partial^2 L}{\partial w \partial v} \Big|_{(0,0)} = \langle \nabla_W V, T \rangle \Big|_a^b + \int_a^b \langle \nabla_T V, \nabla_T W \rangle - \langle R(W,T)T, V \rangle - T\langle V, T \rangle T \langle W, T \rangle.$$

3. Prove Bonnet's theorem that if the sectional curvature K_M of a complete Riemannian manifold M satisfies $K_M \ge 1$ then the diameter of M cannot be bigger than π.

   [*Hint*: Use the index lemma. V, W: vector fields along a geodesic γ from p to q with $V(p) = W(q) = 0$ and $V(q) = W(q)$. If γ has no conjugate points and V is the unique Jacobi field, then $I(V,V) \le I(W,W)$ and equality holds only if $V = W$.]

4. Prove the Rauch comparison theorem:

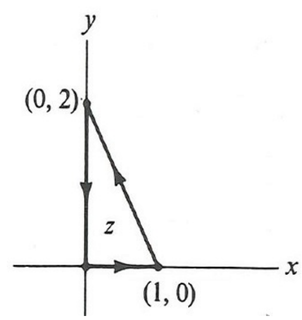
   Assume $K_M \leq a$.

   (a) If J is a Jacobi field along a unit-speed geodesic $\gamma|_{[0,\ell]}$ and J is perpendicular to γ, then $||J||'' + a||J|| \geq 0$ along γ.

   (b) If ψ is a solution on $[0, \ell]$ of $\psi'' + a\psi = 0,\ \psi(0) = ||J||(0),\ \psi'(0) = ||J||'(0)$, and $\psi \neq 0$ on $(0, \ell)$, then $\left(\frac{||J||}{\psi}\right)' \geq 0$ and $||J|| \geq \psi$ on $(0, \ell)$.

5. (a) If α and β are closed differential forms, prove that $\alpha \wedge \beta$ is closed. If, in addition, β is exact, prove that $\alpha \wedge \beta$ is exact.

   (b) Let $\alpha = (e^y + 2xy)dx + (xe^y + x^2 + 1)dy$ on $\mathbb{R}^2$. Show that α is closed. Show that α is exact by finding a function $f : \mathbb{R}^2 \to \mathbb{R}$ with $\alpha = df$. What would the integral of α over the following 1-cycle z be?



6. Determine the de Rham cohomology of $\mathbb{S}^3$.

   [*Hint*: i) Prove that every closed 1-form on $\mathbb{S}^3$ is exact.
   ii) Use the Poincaré duality: $H^{n-p}_{\text{deR}}(M) \cong (H^p_{\text{deR}}(M))^*$.]

7. Use the Stokes' theorem $\int_M d\omega = \int_{\partial M} \omega$ to prove Green's theorem:

   Let $D \subset \mathbb{R}^2$ be a domain with smooth simple closed boundary. If P, Q are smooth functions on D, then

$$\int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \int_{\partial D} Pdx + Qdy,$$

   where D is on the left when we traverse ∂D for the line integral.