This coop original notebook is made of foolscap paper

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# RIEMANNIAN GEOMETRY

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Reference Books

Frank Warner : Foundations of Differentiable Manifolds and Lie Groups

Manfredo do Carmo: Riemannian Geometry

Cheeger & Ebin: Comparison Theorems in Riemannian Geometry

Gallot, Hulin, Lafontaine : Riemannian Geometry

$\mathbb{R}^n \to$ surfaces in $\mathbb{R}^n \to$ abstract space

Definition A topological manifold $M$ of dimension $n$ is a topological space with the following properties:

(i) $M$ is Hausdorff;

(ii) $M$ is locally Euclidean of dimension $n$, i.e., each point of $M$ has a neighborhood which is homeomorphic to an open ball of $\mathbb{R}^n$;

(iii) $M$ has a countable basis of open sets.

* differention on $M$ ? $\longrightarrow$ Need a differentiable structure

* $U \subset \mathbb{R}^n$, open, $f : U \to \mathbb{R}$. $f$ is differentiable of class $C^\infty$ (or smooth) on $U$ if the partial derivatives $\frac{\partial^\alpha f}{\partial r^\alpha}$ exist and are continuous on $U$ for all $\alpha = (\alpha_1, \cdots, \alpha_n)$. $\frac{\partial^\alpha f}{\partial r^\alpha} = \frac{\partial^{[\alpha]} f}{\partial r_1^{\alpha_1} \cdots \partial r_n^{\alpha_n}}$, $\alpha = (\alpha_1, \cdots, \alpha_n)$, $[\alpha] = \sum \alpha_i$.

$f : U \to \mathbb{R}^m$ is differentiable of class $C^\infty$ if each $\underset{i=1,\cdots,m}{f_i = r_i \circ f}$ is $C^\infty$.

* If $\varphi$ is a homeomorphism of a connected open set $U \subset M$ onto an open subset of $\mathbb{R}^n$, $\varphi$ is called a coordinate map, $x_i = r_i \circ \varphi$ are called the coordinate functions, and $(U, \varphi)$ is called a coordinate system. $(U, x_1, \cdots, x_n)$

Definition A differentiable structure $\mathcal{F}$ on a topological manifold $M$ is a collection of coordinate systems $\{(U_\alpha, \varphi_\alpha) : \alpha \in A\}$ satisfying

(a) $\bigcup_{\alpha \in A} U_\alpha = M$.

(b) $\varphi_\alpha \circ \varphi_\beta^{-1}$ is $C^\infty$ for all $\alpha, \beta \in A$.

(c) The collection $\mathcal{F}$ is maximal with respect to (b); that is, if $(U, \varphi)$ is a coordinate system s.t. $\varphi \circ \varphi_\alpha^{-1}$ and $\varphi_\alpha \circ \varphi^{-1}$ are $C^\infty$ for all $\alpha \in A$, then $(U, \varphi) \in \mathcal{F}$.

Definition An n-dimensional differentiable manifold is a pair $(M, \mathcal{F})$ consisting of an n-dimensional topological manifold M together with a differentiable structure $\mathcal{F}$.

ex) $\mathbb{R}^n$: $(\mathbb{R}^n, id)$ , $S^n$: $(S^n - \{n\}, P_n)$, $(S^n - \{s\}, P_s)$ stereographic projections

An open subset $U$ of $(M, \mathcal{F})$

$\mathcal{F}_U = \{ (U_\alpha \cap U, \varphi_\alpha|_{U_\alpha \cap U}) : (U_\alpha, \varphi_\alpha) \in \mathcal{F} \}$

$Gl(n, \mathbb{R})$: general linear group of all $n \times n$ nonsingular real matrices $\subset \mathbb{R}^{n^2}$. $Gl(n, \mathbb{R}) = \det^{-1}(\text{nonzero})$ : open

product manifolds : $(M_1, \mathcal{F}_1)$, $(M_2, \mathcal{F}_2)$

$\mathcal{F} = \{ (U_\alpha \times V_\alpha, \varphi_\alpha \times \psi_\beta) : (U_\alpha, \varphi_\alpha) \in \mathcal{F}_1, (V_\alpha, \psi_\beta) \in \mathcal{F}_2 \}$ $\therefore T^n$.

Definition A continuous map $\psi : M \to N$ is said to be differentiable ($C^\infty$) if $\varphi \circ \psi \circ \tau^{-1}$ is $C^\infty$ for each coordinate map $\tau$ on M and $\varphi$ on N. (Clearly the composition of two differentiable ($C^\infty$) maps is again $C^\infty$. $\psi : M \to N$ is a diffeomorphism if $\psi$ is 1-1, onto, $C^\infty$ and if $\psi^{-1}$ is $C^\infty$.

* Tangent vectors  vector in $\mathbb{R}^n \leftrightarrow$ directional derivative

If $f$ is $C^\infty$ on a nbhd of $p \in \mathbb{R}^n$, then a vector $v = (v_1, \cdots, v_n)$ assigns to $f$ the real number $v(f)$ which is the directional derivative of $f$ in the direction $v$ at $p$. That is,

$$v(f) = v_1 \frac{\partial f}{\partial r_1}\Big|_p + \cdots + v_n \frac{\partial f}{\partial r_n}\Big|_p .$$

A differentiable ($C^\infty$) function $\sigma : (-\varepsilon, \varepsilon) \to M$ is called a $C^\infty$ curve in M. Suppose that $\sigma(0) = p \in M$ and let $C^\infty(M)$ be the set of $C^\infty$ functions on M. The tangent vector to the curve $\sigma$ at $t=0$ is a function $\sigma'(0): C^\infty(M) \to \mathbb{R}$ given by $\sigma'(0)f = \left.\dfrac{d(f\circ\sigma)}{dt}\right|_{t=0}$, $f \in C^\infty(M)$.

A tangent vector at $p$ is the tangent vector at $t=0$ of some curve $\sigma: (-\varepsilon, \varepsilon) \to M$ with $\sigma(0) = p$. $T_pM$: the set of all tangent vectors to $M$ at $p$. $\varphi: (x_1, \cdots, x_n) \mapsto (r_1, \cdots, r_n)$

$$\therefore \sigma'(0)f = \frac{d}{dt}(f\circ\sigma)\Big|_{t=0} = \frac{d}{dt} f(x_1(\sigma(t)), \cdots, x_n(\sigma(t)))\Big|_{t=0}$$
$$= \sum_{i=1}^n x_i'(0)\left(\frac{\partial f}{\partial x_i}\right) = \left(\sum_i x_i'(0)\left(\frac{\partial}{\partial x_i}\right)_p\right) f \qquad \therefore \sigma'(0) = \sum_i x_i'(0)\left(\frac{\partial}{\partial x_i}\right)_p.$$
$$\therefore v = \sum_i v(x_i)\frac{\partial}{\partial x_i}\Big|_p.$$

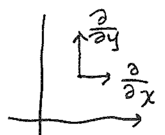
$\therefore$ The tangent to the curve $\sigma$ at $p$ depends only on the derivative of $\sigma$ in a coordinate system.

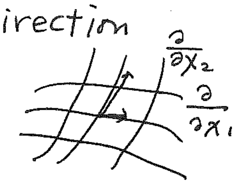
$\therefore$ $T_pM$ forms a vector space of dimension $n$ with a basis $\left\{\left(\frac{\partial}{\partial x_1}\right)_p, \cdots, \left(\frac{\partial}{\partial x_n}\right)_p\right\}$.
The tangent space of $M$ at $p$ $T_pM$ does not depend on the coordinate system. Moreover, $\dim T_pM = \dim M$.

<u>Def</u> A tangent vector $v$ at $p \in M$ is a linear derivation on $C^\infty(M)$, that is, $v(af+bg) = av(f) + bv(g)$ and $v(fg) = f(p)v(g) + g(p)v(f)$, $f, g \in C^\infty(M)$.

<u>Definition</u> $(U, \varphi)$, $x_1, \cdots, x_n$ : coordinate system on $M$.
Define a tangent vector $\left.\frac{\partial}{\partial x_i}\right|_p \in T_pM$ by
$$\left(\frac{\partial}{\partial x_i}\Big|_p\right)(f) = \left.\frac{\partial(f\circ\varphi^{-1})}{\partial r_i}\right|_{\varphi(p)}$$
: the directional derivative of $f$ at $p$ in the $x_i$ coordinate direction

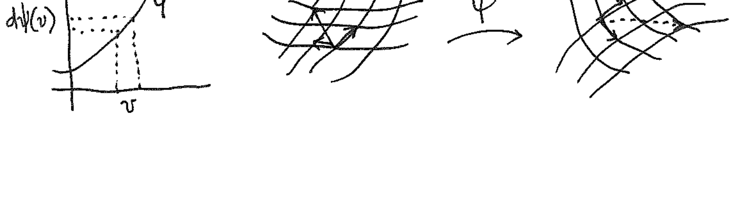
$\left(\frac{\partial}{\partial x_i}\Big|_p\right)(f)$ denotes $\left.\frac{\partial f}{\partial x_i}\right|_p$.




$\therefore$ $\frac{\partial}{\partial x_i}$ depends not only on $x_i$ but all $x_1, \cdots, x_n$.

<u>Definition</u> $\psi: M \to N$, $C^\infty$. $p \in M$. The <u>differential</u> of $\psi$ at $p$ is the linear map $d\psi: T_pM \to T_{\psi(p)}N$ defined by $d\psi(v)(g) = v(g\circ\psi)$, $v \in T_p(M)$, $g \in C^\infty(N)$.
In other words, $\sigma: (-\varepsilon, \varepsilon) \to M$, $\sigma(0) = p$, $\sigma'(0) = v$.
Take $\beta = \psi\circ\sigma$. Then $d\psi(v) = \beta'(0)$.




One can visualize $d\psi$.

* $(U, x_1, \cdots, x_n)$, $(V, y_1, \cdots, y_m)$ : coordinate systems about $p \in M^n$ and $\psi(p) \in N^m$, respectively. Then

$$d\psi\left(\frac{\partial}{\partial x_j}\Big|_p\right) = \sum_{i=1}^{m} \frac{\partial(y_i \circ \psi)}{\partial x_j}\Big|_p \frac{\partial}{\partial y_i}\Big|_{\psi(p)} . \qquad \left\{\frac{\partial(y_i \circ \psi)}{\partial x_j}\right\} : \text{Jacobian of } \psi$$

* Chain rule. $\psi : M \to N$, $\varphi : N \to X$, $C^\infty$ maps. Then

$$d(\varphi \circ \psi)_p = d\varphi_{\psi(p)} \circ d\psi_p \quad \text{or simply} \quad d(\varphi \circ \psi) = d\varphi \circ d\psi .$$

* Tangent bundle $\quad T(M) = \bigcup_{p \in M} T_p(M)$

natural projection : $\pi : T(M) \to M$, $\pi(v) = p$ if $v \in T_pM$.

$(U, \varphi) \in \mathcal{F}$ with coordinate functions $x_1, \cdots, x_n$. $\quad v = \sum a_i \frac{\partial}{\partial x_i}$

Define $\tilde{\varphi} : \pi^{-1}(U) \to \mathbb{R}^{2n}$ by $\quad$ for all $v \in \pi^{-1}(U)$.

$$\tilde{\varphi}(v) = (x_1, \cdots, x_n, a_1, \cdots, a_n) = (x_1(\pi(v)), \cdots, x_n(\pi(v), v(x_1), \cdots, v(x_n)).$$

If $(U, \varphi)$ and $(V, \psi) \in \mathcal{F}$, then $\tilde{\psi} \circ \tilde{\varphi}^{-1}$ is $C^\infty$.

$\{\tilde{\varphi}^{-1}(W) : W \text{ open in } \mathbb{R}^{2n}, (U, \varphi) \in \mathcal{F}\}$ : a basis for a topology on $T(M)$,

$\Rightarrow T(M)$ becomes a $2n$-dimensional, 2nd countable, locally Euclidean.

Let $\tilde{\mathcal{F}}$ be the maximal collection containing $\{(\pi^{-1}(U), \tilde{\varphi}) : (U, \varphi) \in \mathcal{F}\}$.

Then $\tilde{\mathcal{F}}$ is a differentiable structure on $T(M)$.

$T(M)$ with this differentiable structure is called the <u>tangent bundle</u>.

ex) $T(S^1) = S^1 \times \mathbb{R}^1$.

$T(S^2) \neq S^2 \times \mathbb{R}^2 \qquad T(T^2) = T^2 \times \mathbb{R}^2$

<u>Definition</u> Let $\psi : M \to N$ be $C^\infty$.

(a) $\psi$ is an <u>immersion</u> if $d\psi_p$ is nonsingular for each $p \in M$.

(b) If $\psi$ is a 1-1 immersion, then the image $\psi(M)$ is called a <u>submanifold</u>

(c) $\psi$ is an <u>embedding</u> if $\psi$ is a 1-1 immersion which is also a homeomorphism into; that is, $\psi$ is open as a map into $\psi(M)$ with the relative topology.

(d) $\psi$ is a <u>diffeomorphism</u> if $\psi$ maps $M$ 1-1 onto $N$ and $\psi^{-1}$ is $C^\infty$.