$\therefore$ The Jacobi equation: $\nabla_T \nabla_T V = R(T,V)T$. $(*)$

A vector field V, along a geodesic γ with tangent vector T satisfying this equation, is called a Jacobi field.

$\{E_i(t)\}$: orthonormal and parallel along $\alpha(t,0)$.

$(*) \Rightarrow \begin{bmatrix} \langle J,E_1\rangle \\ \vdots \\ \langle J,E_n\rangle \end{bmatrix}'' = \langle R(T,E_i)T, E_j\rangle \begin{bmatrix} \vdots \\ \langle J,E_i\rangle \end{bmatrix}$: linear second order system of ODE's

There exists a unique solution with prescribed initial value and first derivative

: $J(0)$, $J'(0) = \nabla_T J|_{t=0}$: initial conditions

Since $\nabla_T T = 0$, we have $\langle J,T\rangle'' = \langle J'',T\rangle = \langle R(T,J)T,T\rangle = 0$.

$\therefore$ Any Jacobi field J may be written uniquely as $J = J_0 + (at+b)T$, where $\langle J_0, T\rangle \equiv 0$.

* $d\exp(tW)$ arises as the variation field of the 1-parameter family of geodesics $\exp_p \circ \varphi_s$.

  If J is a Jacobi field, then J comes from a variation of geodesics. In fact, let $c(s)$ be a curve such that $c'(0) = J(0)$, and let T, $J'(0)$ be extended to parallel fields along $c(s)$. Then the variation field of $\exp_{c(s)}(t(T + sJ'(0)))$ is a Jacobi field with the same initial conditions as J. Therefore it equals J by uniqueness.

* Taylor series for $\|d\exp(tW)\|^2$.

  Set $d\exp(V) = T$ and $d\exp(tW) = J$. Then $J'(0) = w$.

  $\langle J,J\rangle|_{t=0} = 0$,

  $\langle J,J\rangle'|_{t=0} = 2\langle J,J'\rangle|_{t=0} = 0$,

  $\langle J,J\rangle''|_{t=0} = 2\langle J',J'\rangle|_{t=0} + 2\langle J'',J\rangle|_{t=0} = 2\|W\|^2 = 2$.

  $J''|_{t=0} = R(T,J)T|_{t=0} = 0$.

  $\langle J,J\rangle''' = 6\langle J'',J'\rangle|_{t=0} + 2\langle J''',J\rangle|_{t=0} = 0$.

  $J''' = \nabla_T(R(T,J)T)|_{t=0} = (\nabla_T R)(T,J)T|_{t=0} + R(T,J')T|_{t=0}$

  $\therefore J'''|_{t=0} = R(T,J')T|_{t=0} = R(T,w)T|_{t=0}$.

  $\langle J,J\rangle'''' = 8\langle J''',J'\rangle|_{t=0} + 6\langle J'',J''\rangle|_{t=0} + 2\langle J'''',J\rangle|_{t=0}$

  $= 8\langle R(T,w)T,w\rangle = -8\langle R(w,T)T,w\rangle$.

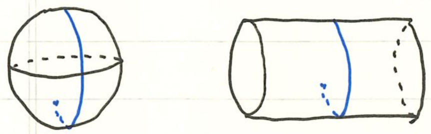
$\therefore\ \|d\exp(tW)\|^2 = t^2 - \frac{1}{3}\langle R(w,T)T, w\rangle t^4 + O(t^5) \quad (*)$

$\|d\exp(tW)\| = t - \frac{1}{6}\langle R(w,T)T, w\rangle t^3 + O(t^4) = \sin t$ on S^2.

$(*) \Rightarrow g_{ij} = \delta_{ij} - \frac{1}{3} R_{ikjl}\, x_k x_l + O(|x|^3)$, $x_1, \cdots, x_n$: normal coordinates.

Conjugate points

geodesic : length = distance (minimizing), not minimizing.



Both not minimizing, but what's the difference?

Infinitely many geodesics between 2 antipodal pts on S^2.

Definition We say that q is conjugate to p if q is a singular value of $\exp: T_pM \to M$. The order of a conjugate point is defined to be the dimension of the null space of $d\exp: T_v(T_pM) \to T_qM$, $\exp(v) = q$.

ex) S^n : order $n-1$.

Pick $v \in T_pM$ and $w \in T_v(T_pM)$ and assume $d\exp(w) = 0$. Then $w \perp v$ because the length of any component of w in the v direction is preserved by $d\exp$.

Proposition q is conjugate to p along a geodesic γ if and only if there exists a nonzero Jacobi field J along γ s.t. $J(0) = J(1) = 0$. Hence q is conjugate to p if and only if p is conjugate to q.

pf) $\rho_s(t) = (v + sw)t$, $\gamma_s = \exp\circ\rho_s$. Then $\frac{d}{ds}(\exp\circ\rho_s(1)) = d\exp(\frac{d}{ds}\rho_s(1)) = 0$.
The 1-parameter family of geodesics γ_s has the variation vector field $\frac{d}{ds}(\gamma_s(t))$ which is a Jacobi field vanishing at $q = \gamma_0(1)$.

$\Leftarrow$) Suppose J exists. $\alpha(t,s) := \exp_{\gamma(0)}(T + sJ'(0))t$. J is the associated variation field. Then $d\exp J'(0)|_{\gamma'(0)\in T_{\gamma(0)}M} = J(1) = 0$. $\therefore$ p is conjugate to q.

Proposition Let $\gamma: [a,b] \to M$ be a geodesic, $\gamma' = T$ and assume that there is a Jacobi field J which vanishes at $\gamma(a)$ and $\gamma(b)$. Then $\langle J, T\rangle \equiv \langle J', T\rangle \equiv 0$.

pf) $T\langle T, J'\rangle = \langle T, J''\rangle = -\langle T, R(J,T)T\rangle = -\langle R(T,T)J, T\rangle = 0$.

$\therefore \langle T, J'\rangle$ is constant. $\langle T, J'\rangle = T\langle T, J\rangle$ $\therefore \langle T, J\rangle$ is linear. $\therefore \langle T, J\rangle \equiv \langle T, J'\rangle \equiv 0$.

$*$ $\langle T, J\rangle = \langle T, J(0)\rangle + \langle T, J'(0)\rangle t$.

<u>Proposition</u> If $\gamma(a)$ and $\gamma(b)$ are not conjugate, then a Jacobi field J along γ is determined by its values at $\gamma(a)$ and $\gamma(b)$.

pf) W, J: Jacobi fields coinciding at $\gamma(a)$, $\gamma(b)$. $\Rightarrow$ $W-J$ is a Jacobi field vanishing at $\gamma(a)$, $\gamma(b)$. Since these points are not conjugate, $W-J\equiv 0$.

Suppose M has constant curvature K. Then

$\langle R(x,y)z,w\rangle = -K(\langle x,z\rangle\langle y,w\rangle - \langle y,z\rangle\langle x,w\rangle)$. $\{E_i\}$: parallel $\Rightarrow$

$\langle J'', E_i\rangle = \langle \nabla_T\nabla_T J, E_i\rangle = \langle R(T,J)T, E_i\rangle = K(\langle J,T\rangle\langle T,E_i\rangle - \langle J,E_i\rangle)$

$\therefore$ If $\langle J,T\rangle = 0$, $\langle J,E_i\rangle'' = -K\langle J,E_i\rangle$

$\therefore$ $K>0$: $J=\sum(a_i\sin(\sqrt{K}t)+b_i\cos(\sqrt{K}t))E_i(t)$.

$K=0$: $\sum(a_it+b_i)E_i(t)$.

$K<0$: $\sum(a_i\sinh(\sqrt{-K}t)+b_i\cosh(\sqrt{-K}t))E_i(t)$.

* If $K\le 0$, geodesics have no conjugate points, while if $K>0$ conjugate points occur at $t=\dfrac{l\pi}{\sqrt{K}}$, $l=1,2,3,\cdots$.

<u>Second Variation of Arc Length</u>

geodesic : critical pt of arclength function. minimizing, not local minimum

$\gamma:[a,b]\to M$ geodesic. $\alpha: Q\to M$, C^∞, $Q=[a,b]\times(-\varepsilon,\varepsilon)\times(-\delta,\delta)$, $\alpha(t,0,0)=\gamma(t)$.

$L(v,w)=\int_a^b \|T\|ds$: the arc length of the curve $t\mapsto\alpha(t,v,w)$.

Assume $\|\gamma'\|\equiv 1$. Let $T=d\alpha(\frac{\partial}{\partial t})$, $V=d\alpha(\frac{\partial}{\partial v})$, $W=d\alpha(\frac{\partial}{\partial w})$.

$\frac{\partial}{\partial v}L(v,w)=\int_a^b\frac{\langle\nabla_TV,T\rangle}{\|T\|}$. Then $\frac{\partial^2}{\partial w\partial v}L(v,w)=\frac{\partial}{\partial w}\int_a^b\frac{\langle\nabla_TV,T\rangle}{\|T\|}$

$$=\int_a^b\frac{\langle\nabla_W\nabla_TV,T\rangle+\langle\nabla_TV,\nabla_WT\rangle}{\|T\|}-\langle\nabla_TV,T\rangle\frac{\langle\nabla_WT,T\rangle}{\|T\|^3}$$

$$=\int_a^b\frac{\langle R(W,T)V,T\rangle+\langle\nabla_T\nabla_WV,T\rangle+\langle\nabla_TV,\nabla_TW\rangle}{\|T\|}-\frac{\langle\nabla_TV,T\rangle\langle\nabla_TW,T\rangle}{\|T\|^3}.$$

Using $\|T\|_{(0,0)}\equiv 1$ and $\nabla_TT|_{(0,0)}\equiv 0$,

$$\left.\frac{\partial^2L}{\partial w\partial v}\right|_{(0,0)}=\int_a^b\langle\nabla_TV,\nabla_TW\rangle-\langle R(W,T)T,V\rangle+T\langle\nabla_WV,T\rangle-T\langle V,T\rangle T\langle W,T\rangle$$

$$=\langle\nabla_WV,T\rangle\Big|_a^b+\int_a^b\langle\nabla_TV,\nabla_TW\rangle-\langle R(W,T)T,V\rangle-T\langle V,T\rangle T\langle W,T\rangle$$

: The second variation formula. Valid for piecewise smooth variations. In case the variation is through geodesics, $T\langle V,T\rangle$ and $T\langle W,T\rangle$ are constant. Then if $\langle V,T\rangle$ or $\langle W,T\rangle$ vanishes at both endpoints, the last term drops out. If V or W vanishes at the endpoints, or more generally, $\nabla_V W=0$, we get

$$\left.\frac{\partial^2 L}{\partial w\,\partial v}\right|_{(0,0)} = \int_a^b \langle \nabla_T V, \nabla_T W\rangle + \langle R(W,T)V,T\rangle.$$

In this case the second variation depends only on the restrictions of V, W to γ.
We call the above integral the <u>index form</u> $I(V,W)$. : symmetric bilinear form on the space of piecewise smooth vector fields V, W along γ s.t. $\langle V,T\rangle \equiv \langle W,T\rangle \equiv 0$.
I is independent of the orientation of γ.
If I is positive definite on vector fields vanishing at $\gamma(a)$, $\gamma(b)$, then γ is a minimum among all nearby curves with the same endpoints.

<u>Proposition</u> Let I be defined on all piecewise smooth vector fields along γ which vanish at the endpoints. Then the null space of I is exactly the set of Jacobi fields along γ which vanish at $\gamma(a)$ and $\gamma(b)$. Specifically, V is a Jacobi field if and only if $I(V,W)=0$ for all W.

pf) Let f be a function vanishing at the endpoints and positive elsewhere.
$W := f(t)(-\nabla_T\nabla_T V + R(T,V)T) \Rightarrow V$ is a Jacobi field.

<u>Corollary</u> I has a nontrivial null space if and only if $\gamma(a)$ is conjugate to $\gamma(b)$ along γ. The dimension of the null space is the order of the conjugate point $\gamma(b)$.

pf) $J'(0) \longleftrightarrow \exp_{\gamma(0)}(T+sJ'(0))t$: linear isomorphism between the null space of $d\exp$ and the space of Jacobi fields along γ which vanish at the endpoints.