Ric(v,v) is (n-1) times the average value of the sectional curvature, taken over all the 2-planes containing v.

The scalar curvature is n(n-1) times the average value of the sectional curvature at a given point.

$$R_{ij} = Ric\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \sum_k R^k_{kij} = \sum_{kl} R_{kijl}\, g^{lk}$$

$$S = \sum R_{ij}\, g^{ij}$$

$$g_{ij} = \delta_{ij} - \frac{1}{3} R_{ikjl}\, x_k x_l + O(|x|^3)$$

$$d\mu_g = \left[1 - \frac{1}{6} R_{jk}\, x_j x_k + O(|x|^3)\right] d\mu_{Euclidean}, \quad d\mu_g = \sqrt{\det(g_{ij})}\, dx_1 \cdots dx_n$$

Ric(v,v) represents the amount by which the volume of a narrow conical piece in v direction deviates from that in $\mathbb{R}^n$.

$$Vol(B_r(p) \subset M) = \left[1 - \frac{S}{6(n+2)} r^2 + O(r^4)\right] Vol(B_r(0) \subset \mathbb{R}^n)$$

ex) $Vol(B_r(p) \subset S^3) = 4\pi \int_0^r \sin^2 t\, dt = \frac{4\pi}{3} r^3 \left(1 - \frac{r^2}{5} + \frac{2}{105} r^4 - \cdots\right)$, $S \equiv 6$.

Gauss-Bonnet Theorem $\int_S K\, dA = 2\pi \chi(S)$.

$\therefore$ T^2 cannot have positive Gaussian curvature.

1. T^n has no metric with positive scalar curvature. (Gromov-Lawson)
2. Hopf conjecture: $S^2 \times S^2$ cannot admit a metric with positive sectional curv.
3. Every compact manifold admits a metric with negative constant scalar curvature. (Aubin)
4. Every compact M^3 has a metric of negative Ricci curvature. (Gau-Yau)
5. If compact M has positive Ricci curvature, then its first betti number β_1 is zero. (Bochner) Rank $H_q(M) := \beta_q$.
6. A complete noncompact M^3 with positive Ricci curvature is diffeomorphic to $\mathbb{R}^3$.

Tensors on Riemannian manifolds

tensor: natural generalization of a vector field. ex) metric, curvature

Analogously to vector fields, tensors can be differentiated covariantly.

Definition A tensor T of order r on a Riemannian manifold M is a $C^\infty(M)$-multilinear mapping $T: \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{r\ factors} \to C^\infty(M)$.

$\therefore T(Y_1, \cdots, fX+gY, \cdots, Y_r) = f\,T(Y_1, \cdots, X, \cdots, Y_r) + g\,T(Y_1, \cdots, Y, \cdots, Y_r)$.

$C^\infty(M)$-multilinear $\Rightarrow$ We only need $y_1, \cdots, y_r \in T_pM$ to evaluate $T(y_1, \cdots, y_r)$.

* $X \in \mathfrak{X}(M)$. Define $\omega(Y) = \langle X, Y \rangle$ for all $Y \in \mathfrak{X}(M)$.

Then ω is a tensor of order 1. ω is called a 1-form on M.

$x_1, \cdots, x_n$: local coordinates on $M \Rightarrow dx_1, \cdots, dx_n$ form a basis for the set $E^1(M)$ of 1-forms on M. $dx_i(Y) = Y(x_i) \in C^\infty(M)$.

Suppose $X \in \mathfrak{X}(M)$ defines a 1-form ω by $\omega(Y) = \langle X, Y \rangle$, $\forall\, Y \in \mathfrak{X}(M)$.

$\omega = \sum_i \omega_i\, dx_i$, $X = \sum_i X^i \frac{\partial}{\partial x_i}$. $\omega_j = \omega(\frac{\partial}{\partial x_j}) = \langle \sum X^i \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \rangle = \sum_i X^i g_{ij}$.

Denote $\omega = X^\flat$: $X \mapsto \omega$: isomorphism

The inverse isomorphism $\omega \mapsto X = \omega^\#$ is given in local coordinates by

$(\omega^\#)^j = \sum_i g^{ji} \omega_i$.

Definition Let T be a tensor of order r. The covariant derivative ∇T of T is a tensor of order $r+1$ given by

$$\nabla T(Y_1, \cdots, Y_r, Z) = Z(T(Y_1, \cdots, Y_r)) - T(\nabla_Z Y_1, \cdots, Y_r) - \cdots - T(Y_1, \cdots, Y_{r-1}, \nabla_Z Y_r).$$

For each $Z \in \mathfrak{X}(M)$, the covariant derivative $\nabla_Z T$ of T relative to Z is a tensor of order r given by $\nabla_Z T(Y_1, \cdots, Y_r) = \nabla T(Y_1, \cdots, Y_r, Z)$.

For $p \in M$, let $\alpha: (-\varepsilon, \varepsilon) \to M$ be a curve with $\alpha(0) = p$, $\alpha'(t) = Z(\alpha(t))$.

Let $\{e_1, \cdots, e_n\}$ be a basis of T_pM and let $e_i(t)$ be the parallel transport of e_i along $\alpha = \alpha(t)$, for $i = 1, \cdots, n$. Then

$$(\nabla_Z T)(e_{i_1}, \cdots, e_{i_r}) = Z(\alpha(t))\, T(e_{i_1}, \cdots, e_{i_r}) - T(\nabla_Z e_{i_1}, \cdots, e_{i_r}) - \cdots - T(e_{i_1}, \cdots, \nabla_Z e_{i_r})$$
$$= \frac{d}{dt} T(e_{i_1}, \cdots, e_{i_r})$$

* $g(X,Y) = \langle X, Y \rangle \Rightarrow \nabla g(X, Y, Z) = Z\langle X, Y \rangle - \langle \nabla_Z X, Y \rangle - \langle X, \nabla_Z Y \rangle = 0$.

On constant curvature manifold M, $R(u, v, w, x) = K(\langle u, x \rangle \langle v, w \rangle - \langle u, w \rangle \langle v, x \rangle)$.

$\therefore \nabla R = 0$

A Riemannian manifold M is called locally symmetric if for each $p \in M$ there exists r s.t. reflection through the origin (in normal coordinates) is an isometry on $B_r(p)$. M is locally symmetric if and only if $\nabla R = 0$.

* $f \in C^\infty(M) \Rightarrow f: M \to \mathbb{R}$, df is a 1-form on M. What is the vector field X corresponding to df? i.e., $df(Y) = \langle X, Y \rangle = \langle \nabla f, Y \rangle$. ∇f: gradient of f.

Given a 1-form $\omega = \sum \omega_i dx_i$, what is its dual vector field $X = \sum X^i \frac{\partial}{\partial x_i}$?

$\omega(Y) = \langle X, Y \rangle$. $\therefore \omega(\frac{\partial}{\partial x_k}) = \langle \sum X^i \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_k} \rangle$, $\omega_k = \sum X^i g_{ik}$,

$\sum \omega_k g^{kl} = \sum X^i g_{ik} g^{kl} = \sum X^i \delta_i^{\,l} = X^l$. $\therefore X = \sum \omega_k g^{ki} \frac{\partial}{\partial x_i} = \sum \omega_j g^{ji} \frac{\partial}{\partial x_i}$

$\therefore X^i = \sum_j \omega_j g^{ji}$

$df = \sum \frac{\partial f}{\partial x_i} dx_i$ $\therefore \nabla f = \sum \frac{\partial f}{\partial x_j} g^{ji} \frac{\partial}{\partial x_i}$

Hessian $\nabla^2 f = \nabla df$: tensor of order 2.

$\nabla^2 f(X,Y) = (\nabla_Y df)(X) = Y(df(X)) - df(\nabla_Y X) = YXf - (\nabla_Y X)f$

$\nabla^2 f(X,Y) - \nabla^2 f(Y,X) = YXf - XYf - (\nabla_Y X)f + (\nabla_X Y)f = ([Y,X] - (\nabla_Y X - \nabla_X Y))f$

$= 0$. $\therefore \nabla^2 f$ is a symmetric tensor.

$\Delta f = tr(\nabla^2 f) = \sum_i \nabla^2 f(e_i, e_i)$, $\{e_i\}$: orthonormal basis.

Given $X \in \mathfrak{X}(M)$, the divergence of X is defined to be a function $div X \in C^\infty(M)$ as the trace of the linear map $Y \to \nabla_Y X$. $\therefore div X = \sum dx_i(\nabla_{\frac{\partial}{\partial x_i}} X)$,

$div X = \sum \langle \nabla_{e_i} X, e_i \rangle$, $\{e_i\}$: orthonormal basis.

Use normal coordinates $x_1, \cdots, x_n$ in a nbhd of p. $\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \rangle = \delta_{ij}$, $\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = 0$ at p.

$\therefore \Delta f = \sum e_i e_i f - (\nabla_{e_i} e_i) f = \sum \frac{\partial}{\partial x_i} \frac{\partial f}{\partial x_i}$.

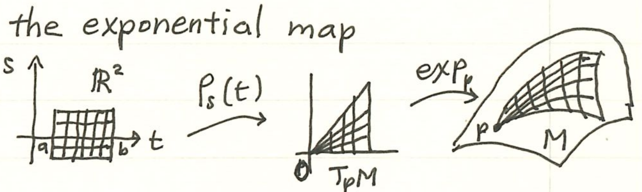
$\nabla f = \sum \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_i}$, $div \nabla f = \sum_{ij} \langle \nabla_{\frac{\partial}{\partial x_j}} (\frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_i}), \frac{\partial}{\partial x_j} \rangle = \sum \frac{\partial}{\partial x_i} \frac{\partial f}{\partial x_i}$ $\therefore \Delta f = div \nabla f$.

Relationship between curvature and the exponential map

v, w : orthonormal in T_pM.

$\rho_s(t) = (v + sw)t$: family ρ_s of rays in T_pM

$\exp_p \circ \rho_s$: geodesic through p with initial tangent vector $v + sw$



v and w induce parallel vector fields V and W on T_pM.

The effect of $d\exp_p$ on vectors of T_pM ? Compute the Taylor expansion of $\|d\exp(tW)\|^2$

$d\exp(tW)$ arises as the variation field of the 1-parameter family of geodesics $\exp_p \circ \rho_s$.

Fields of this type are called Jacobi fields.

$\alpha := \exp_p \circ \rho_s : [a,b] \times (-\varepsilon, \varepsilon) \to M$. For fixed s, $\alpha(t,s)$ is a geodesic.

Let $T = d\alpha(\frac{\partial}{\partial t})$, $V = d\alpha(\frac{\partial}{\partial s})$. Differential equation of $V|_{\alpha(t,0)}$?

$\nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial s} - \nabla_{\frac{\partial}{\partial s}} \frac{\partial}{\partial t} - [\frac{\partial}{\partial t}, \frac{\partial}{\partial s}] = 0$ $\therefore \nabla_T V - \nabla_V T - d\alpha [\frac{\partial}{\partial t}, \frac{\partial}{\partial s}] = 0$

$\therefore \nabla_T V = \nabla_V T$, $\nabla_T \nabla_T V = \nabla_T \nabla_V T = \nabla_T \nabla_V T - \nabla_V \nabla_T T$ ($\because \nabla_T T = 0$.)

$= R(T,V)T$ ($\because [T,V] = 0$.)