The simplest examples of Riemannian manifolds are those whose sectional curvature is constant. The complete ones are called <u>space forms</u>. For each K all simply connected space forms with sectional curvature K are isometric.

$x_1, \cdots, x_n$: standard coordinates of $\mathbb{R}^n$, $g_{ij} = \dfrac{\delta_{ij}}{(1+\frac{K}{4}\sum x_i^2)^2}$

$(\mathbb{R}^n, g_{ij}) = S^n_{\frac{1}{\sqrt{K}}}$: constant curvature $K>0$, not complete.

$(\|x\|^2 < -\frac{4}{K}, g_{ij})$: $K<0$.

Define $\langle R(u,v)w, x\rangle = -K \begin{vmatrix} \langle u,w\rangle & \langle v,w\rangle \\ \langle u,x\rangle & \langle v,x\rangle \end{vmatrix}$: well-defined, i.e, (**).

Moreover, R has constant sectional curvature K.

$$R(u,v)w = K(\langle w,v\rangle u - \langle w,u\rangle v).$$

$x_1, \cdots, x_n$: normal coordinates in a neighborhood of $p \in M$.

$g_{ij} = \delta_{ij} - \frac{1}{3}R_{ikjl}x_kx_l + O(|x|^3)$. $\because \Gamma^k_{ij} = \frac{1}{2}\sum_m\left(\frac{\partial}{\partial x_i}g_{jm} + \frac{\partial}{\partial x_j}g_{mi} - \frac{\partial}{\partial x_m}g_{ij}\right)g^{mk}$

$\Gamma^k_{ij}(p) = 0$, $\frac{\partial}{\partial x_k}g_{ij}(p) = 0$.

<u>Definition</u> M^n is said to have <u>isotropic curvature</u> at p if $K(\sigma)$ is independent of the choice of $\sigma \subset T_pM$. M is <u>isotropic</u> if it is isotropic at every point. $n \geq 3$.

Lemma $\Rightarrow$ $R(u,v,w,x) = f(\langle u,w\rangle\langle v,x\rangle - \langle u,x\rangle\langle v,w\rangle)$, $f \in C^\infty(M)$.

<u>Theorem (Schur)</u> An isotropic Riemannian manifold M^n, $n \geq 3$, has constant curvature.

<u>Ricci curvature and Scalar curvature</u>

<u>Def</u> The <u>Ricci curvature</u> is a symmetric bilinear form on T_pM for each $p \in M$ defined to be the trace of the linear transformation $z \mapsto R(z,x)y$.

Hence $Ric(x,y) = \sum_i \langle R(e_i,x)y, e_i\rangle$, where $\{e_i\}$ is an orthonormal basis of T_pM.

symmetric $\because$ $Ric(x,y) = \sum\langle R(e_i,x)y,e_i\rangle = \sum\langle R(y,e_i)e_i,x\rangle = Ric(y,x)$

The <u>Scalar curvature</u> is the trace of the Ricci curvature.

$$S = \sum_{ij}\langle R(e_i,e_j)e_j, e_i\rangle$$

Ric : symmetric $\Rightarrow$ Ricci curvature is completely determined by the quantity $Ric(v,v)$ for all vectors of unit length.

$Ric(v,v)$ is $(n-1)$ times the average value of the sectional curvature, taken over all the 2-planes containing v.
The scalar curvature is $n(n-1)$ times the average value of the sectional curvature at a given point.

$$R_{ij} = Ric\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \sum_k R^k_{kij} = \sum_{kl} R_{kijl}\, g^{lk}$$

$$S = \sum R_{ij}\, g^{ij}$$

$$g_{ij} = \delta_{ij} - \frac{1}{3} R_{ikjl}\, x_k x_l + O(|x|^3)$$

$$d\mu_g = \left[1 - \frac{1}{6} R_{jk}\, x_j x_k + O(|x|^3)\right] d\mu_{Euclidean}, \quad d\mu_g = \sqrt{\det(g_{ij})}\, dx_1 \cdots dx_n$$

$Ric(v,v)$ represents the amount by which the volume of a narrow conical piece in v direction deviates from that in $\mathbb{R}^n$.

$$Vol(B_r(p) \subset M) = \left[1 - \frac{S}{6(n+2)} r^2 + O(r^4)\right] Vol(B_r(0) \subset \mathbb{R}^n)$$

ex) $Vol(B_r(p) \subset S^3) = 4\pi \int_0^r \sin^2 t\, dt = \frac{4\pi}{3} r^3 \left(1 - \frac{r^2}{5} + \frac{2}{105} r^4 - \cdots\right)$, $S \equiv 6$.

Gauss-Bonnet Theorem $\int_S K\, dA = 2\pi \chi(S)$.
$\therefore$ T^2 cannot have positive Gaussian curvature.

1. T^n has no metric with positive scalar curvature. (Gromov-Lawson)
2. Hopf conjecture: $S^2 \times S^2$ cannot admit a metric with positive sectional curv.
3. Every compact manifold admits a metric with negative constant scalar curvature. (Aubin)
4. Every compact M^3 has a metric of negative Ricci curvature. (Gau-Yau)
5. If compact M has positive Ricci curvature, then its first betti number β_1 is zero. (Bochner) Rank $H_q(M) := \beta_q$.
6. A complete noncompact M^3 with positive Ricci curvature is diffeomorphic to $\mathbb{R}^3$.