<u>Theorem</u> (Hopf - Rinow) The following are equivalent:

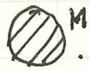
(a) M is a complete metric space where the distance from p to q in M is defined as the minimum length of all curves from p to q.

(b) For some $p \in M$, $\exp_p$ is defined on all of T_pM.

(c) For all $p \in M$, $\exp_p$ is defined on all of T_pM.

Any of these conditions imply

(d) Any two points p, q of M can be joined by a geodesic whose length is the distance from p to q.

(a) $\Rightarrow$ (d) most important. M: compact $\Rightarrow$ (a). (d) $\not\Rightarrow$ (a) $\because$ 

$\boxed{\text{CURVATURE}}$

Gauss : curvature of a surface. intrinsic

Riemann : curvature in a Riemannian manifold M.

$p \in M$, σ: 2-dimensional subspace of T_pM

$\exp(\sigma)$: the set of all geodesics that start at p and are tangent to σ.

: surface in M. its own curvature as defined by Gauss.

<u>Definition</u> The <u>curvature</u> R of a Riemannian manifold M is a trilinear map from $\mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M)$ to $\mathfrak{X}(M)$ given by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \quad X, Y, Z \in \mathfrak{X}(M).$$

* For the natural coordinates of $\mathbb{R}^n$ $x_1, \cdots, x_n$, let $Z = \sum_i z_i \frac{\partial}{\partial x_i}$.

Then $\nabla_X \nabla_Y Z = (XYz_1, \cdots, XYz_n)$ and hence

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z = 0.$$

$\therefore$ R measures how much M deviates from being Euclidean.

* If $x_1, \cdots, x_n$ are coordinates of a Riemannian manifold M, then $\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = 0$.

$\therefore$ $R\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)\frac{\partial}{\partial x_k} = \left(\nabla_{\frac{\partial}{\partial x_i}} \nabla_{\frac{\partial}{\partial x_j}} - \nabla_{\frac{\partial}{\partial x_j}} \nabla_{\frac{\partial}{\partial x_i}}\right)\frac{\partial}{\partial x_k}$, hence the curvature measures the non-commutativity of the covariant derivative.

* exp preserves the radial distance and the orthogonality between $\gamma_v'(t)$ and $d\exp(w)$. Hence the deviation of $d\exp$ from being an

isometry is measured by the extent to which $\|w\|$ differs from $\|d\exp(w)\|$.

$$\|d\exp(tw)\|^2 = \|tw\|^2 - \frac{1}{3}\langle R(tw,T)T, tw\rangle t^2 + O(t^5).$$

* R is $C^\infty(M)$-linear. $\therefore$ tensor

$$R(fX,Y)Z = fR(X,Y)Z = R(X,fY)Z = R(X,Y)fZ.$$

* $$\left.\begin{aligned} &R(X,Y)Z + R(Z,X)Y + R(Y,Z)X = 0 \\ &R(X,Y)Z = -R(Y,X)Z. \\ &\langle R(X,Y)Z, W\rangle = -\langle R(X,Y)W, Z\rangle. \\ &\langle R(X,Y)Z, W\rangle = \langle R(Z,W)X, Y\rangle. \end{aligned}\right\} \begin{aligned} &\text{Bianchi identity} \\ &(**) \end{aligned}$$

In coordinate system:

$$R\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)\frac{\partial}{\partial x_k} = \sum_l R^l_{ijk}\frac{\partial}{\partial x_l} = \nabla_{\frac{\partial}{\partial x_i}}\nabla_{\frac{\partial}{\partial x_j}}\frac{\partial}{\partial x_k} - \nabla_{\frac{\partial}{\partial x_j}}\nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_k} = \nabla_{\frac{\partial}{\partial x_i}}\Big(\sum_l \Gamma^l_{jk}\frac{\partial}{\partial x_l}\Big) - \nabla_{\frac{\partial}{\partial x_j}}\Big(\sum_l \Gamma^l_{ik}\frac{\partial}{\partial x_l}\Big)$$

$$R\Big(\sum_i u^i\frac{\partial}{\partial x_i}, \sum_j v^j\frac{\partial}{\partial x_j}\Big)\sum_k w^k\frac{\partial}{\partial x_k} = \sum_{ijkl} R^l_{ijk} u^i v^j w^k \frac{\partial}{\partial x_l}$$

$$\therefore R^m_{ijk} = \sum_l \Gamma^l_{jk}\Gamma^m_{il} - \sum_l \Gamma^l_{ik}\Gamma^m_{jl} + \frac{\partial}{\partial x_i}\Gamma^m_{jk} - \frac{\partial}{\partial x_j}\Gamma^m_{ik}.$$

Set $\left\langle R\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_m}\right\rangle = \sum_l R^l_{ijk} g_{lm} := R_{ijkm}$. Then

$$R_{ijkm} + R_{jkim} + R_{kijm} = 0, \quad R_{ijkm} = -R_{jikm}, \quad R_{ijkm} = -R_{ijmk}, \quad R_{ijkm} = R_{kmij}.$$

* Sectional Curvature

Given any plane σ in T_pM and two vectors v and w which span σ, we define the <u>sectional curvature</u> $K(\sigma)$ to be $\frac{\langle R(v,w)w, v\rangle}{\|v\wedge w\|^2}$, where $\|v\wedge w\|$ denotes the area of the parallelogram spanned by v and w.

$K(\sigma)$ does not depend on the choice of the spanning vectors.

$\because$ $\{v,w\} \to \{w,v\}$ $\quad\{v,w\} \to \{\lambda v, w\}$ $\quad\{v,w\} \to \{v+\lambda w, w\}$

The curvature tensor R is completely determined by knowledge of $K(\sigma)$ for all σ:

<u>Lemma</u> Let R, R' be trilinear maps from $\mathfrak{X}(M)\times\mathfrak{X}(M)\times\mathfrak{X}(M)$ to $\mathfrak{X}(M)$ such that conditions $(**)$ are satisfied. If $\langle R(v,w)w,v\rangle = \langle R'(v,w)w,v\rangle$ for all v, w, then $R = R'$.

pf) $\langle R(u+w, v)(u+w), v\rangle = \langle R'(u+w, v)(u+w), v\rangle$ $\quad\therefore \langle R(u,v)w,v\rangle = \langle R'(u,v)w,v\rangle$

$$\langle R(u, v+x)w, v+x\rangle = \langle R'(u, v+x)w, v+x\rangle$$

$$\begin{aligned}\therefore \langle R(u,v)w,x\rangle - \langle R'(u,v)w,x\rangle &= \langle R(v,w)u,x\rangle - \langle R'(v,w)u,x\rangle \\ &= \langle R(w,u)v,x\rangle - \langle R'(w,u)v,x\rangle\end{aligned}$$

$\therefore$ Bianchi $\Rightarrow \langle R(u,v)w,x\rangle = \langle R'(u,v)w,x\rangle$.

The simplest examples of Riemannian manifolds are those whose sectional curvature is constant. The complete ones are called <u>space forms</u>. For each K all simply connected space forms with sectional curvature K are isometric.

$x_1, \cdots, x_n$: standard coordinates of $\mathbb{R}^n$, $g_{ij} = \dfrac{\delta_{ij}}{(1+\frac{K}{4}\sum x_i^2)^2}$

$(\mathbb{R}^n, g_{ij}) = S^n_{\frac{1}{\sqrt{K}}}$: constant curvature $K>0$, not complete.

$(\|x\|^2 < -\frac{4}{K}, g_{ij})$: $K<0$.

Define $\langle R(u,v)w, x\rangle = -K \begin{vmatrix} \langle u,w\rangle & \langle v,w\rangle \\ \langle u,x\rangle & \langle v,x\rangle \end{vmatrix}$: well-defined, i.e, (**).

Moreover, R has constant sectional curvature K.

$$R(u,v)w = K(\langle w,v\rangle u - \langle w,u\rangle v).$$

$x_1, \cdots, x_n$: normal coordinates in a neighborhood of $p \in M$.

$g_{ij} = \delta_{ij} - \frac{1}{3} R_{ikjl} x_k x_l + O(|x|^3)$. $\because \Gamma^k_{ij} = \frac{1}{2}\sum_m \left(\frac{\partial}{\partial x_i} g_{jm} + \frac{\partial}{\partial x_j} g_{mi} - \frac{\partial}{\partial x_m} g_{ij}\right) g^{mk}$

$\Gamma^k_{ij}(p) = 0$, $\frac{\partial}{\partial x_k} g_{ij}(p) = 0$.

<u>Definition</u> M^n is said to have <u>isotropic curvature</u> at p if $K(\sigma)$ is independent of the choice of $\sigma \subset T_pM$. M is <u>isotropic</u> if it is isotropic at every point. $n \geq 3$.

Lemma $\Rightarrow$ $R(u,v,w,x) = f(\langle u,w\rangle\langle v,x\rangle - \langle u,x\rangle\langle v,w\rangle)$, $f \in C^\infty(M)$.

<u>Theorem</u> (Schur) An isotropic Riemannian manifold M^n, $n \geq 3$, has constant curvature.