$= \frac{1}{2}\int_a^b \langle T,T\rangle^{-\frac{1}{2}} V\langle T,T\rangle dt = \int_a^b \langle T,T\rangle^{-\frac{1}{2}} \langle \nabla_V T, T\rangle dt.$

Since $[T,V]=0$ on Q and $\nabla_T V - \nabla_V T = [T,V]$,

$= \int_a^b \langle T,T\rangle^{-\frac{1}{2}} \langle \nabla_T V, T\rangle dt.$ Since $\|c_0'\| = \ell$,

$\frac{d}{ds} L[c_s]|_{s=0} = \frac{1}{\ell}\int_a^b \langle \nabla_T V, T\rangle dt = \frac{1}{\ell}\int_a^b \left(T\langle V,T\rangle - \langle V, \nabla_T T\rangle\right) dt.$

$= \frac{1}{\ell}\left\{ \langle V,T\rangle|_a^b - \int_a^b \langle V, \nabla_T T\rangle dt \right\}$: the 1st variation formula

If all curves c_s have the same endpoints, then $V(a,0) = V(b,0) = 0$.

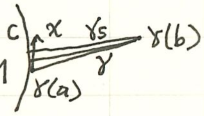
$\therefore \frac{d}{ds} L[c_s]|_{s=0} = -\frac{1}{\ell}\int_a^b \langle V, \nabla_T T\rangle dt.$

If $c = c_0$ is the shortest curve from $c(a)$ to $c(b)$, then $\frac{d}{ds} L[c_s]|_{s=0} = 0$ for any α. $V := \varphi(t)\nabla_T T$, $\varphi(t) > 0$, $\varphi(a) = \varphi(b) = 0 \Rightarrow \nabla_T T = \nabla_{c'} c' \equiv 0$.

$\therefore$ shortest curve $\underset{\Longleftarrow \text{ only locally}}{\Longrightarrow}$ geodesic. How about $\Leftarrow$?

<u>Proposition</u> $N, \bar{N}$: two submanifolds of M, without boundary.
Let $\gamma : [a,b] \to M$ be a geodesic s.t. $\gamma(a) \in N$, $\gamma(b) \in \bar{N}$ and γ is the shortest curve from N to $\bar{N}$. Then $\gamma'(a)$ is perpendicular to $T_{\gamma(a)}N$ and $\gamma'(b)$ is perpendicular to $T_{\gamma(b)}N$

pf) If $\gamma'(a)$ is not perpendicular to $T_{\gamma(a)}N$, choose $x \in T_{\gamma(a)}N$ s.t. $\langle \gamma'(a), x\rangle > 0$, and let c be a curve in N starting at $\gamma(a)$ s.t. $c'(0) = x$. Construct a variation $\alpha : [a,b]\times(-\varepsilon,\varepsilon) \to M$



s.t. $\alpha|_{[a,b]\times\{0\}} = \gamma$, $\alpha(a,s) = c(s)$, $\alpha(b,s) \equiv \gamma(b)$.

Then if $\gamma_s = \alpha|_{[a,b]\times\{s\}}$, 1st Var. Form. shows that

$\frac{d}{ds} L[\gamma_s]|_{s=0} = -\frac{1}{\ell}\langle \gamma'(0), x\rangle < 0.$ $\therefore \gamma$ is not minimal. Same at $\gamma(b)$.

M : Riemannian manifold $\Rightarrow T_pM$ is equipped with an innerproduct.
$\forall v \in T_pM$, $T_v(T_pM)$ can be identified with T_pM. $\therefore T_v(T_pM)$ inherits an inner product. $d\exp : T_v(T_pM) \to T_{\exp(v)}M$ does not preserve the inner product but preserves the length of $\rho'(t)$ where $\rho(t) = tv$ is the ray from $0 \in T_pM$ through v. $\exp(\rho(t)) = \gamma_v(t)$, $d\exp(\rho'(t)) = \gamma_v'(t)$,
$\|\rho'(t)\| = \|\gamma_v'(t)\|$.

Gauss Lemma If $\rho(t)=tv$ is a ray through the origin in T_pM and $w\in T_{\rho(1)}(T_pM)$ is perpendicular to $\rho'(1)$, then $d\exp(w)$ is perpendicular to $d\exp(\rho'(1))$.

pf) Let $c(s)$ be a curve in T_pM s.t. $c(0)=v$, $c'(0)=w$, $dist(0,c(s))=\|v\|=\ell$

Define $\alpha(t,s)=\exp(\rho_s(t))$, where $\rho_s [0,1]\to T_pM$ is the ray from 0 to $c(s)$ in T_pM.

The lengths of the geodesics $t\mapsto\alpha(t,s)$ are independent of s and

$d\alpha\frac{\partial}{\partial s}(0,0)=0$, $d\alpha\frac{\partial}{\partial s}(1,0)=d\exp(w)$. The first variation formula ⇒

$$0=\frac{d}{ds}L[\exp(\rho_s)]\Big|_{s=0}=\frac{1}{\ell}\langle V,T\rangle\Big|_0^1=\frac{1}{\ell}\langle d\alpha\frac{\partial}{\partial s},\gamma_v'(t)\rangle\Big|_0^1=\frac{1}{\ell}\langle d\exp(w),d\exp(\rho'(1))\rangle.$$

$S_p(r)$: sphere of radius r centered at p in T_pM. ⇒ $\gamma_v(t)$ is orthogonal to $\exp_p(S_p(r))$

Let $\theta_1,\cdots,\theta_{n-1}$ be coordinates on $S_p(r)$. ⇒ $r,\theta_1,\cdots,\theta_{n-1}$: coordinates on $T_pM-\{0\}$.

exp : rays through the origin → geodesics through p. ⇒ $r,\theta_1,\cdots,\theta_{n-1}$: coord. on $M\setminus\{p\}$

Metric on $M\setminus\{p\}$: $g=a_{11}\,dr^2+\sum_{i=1}^{n-1}a_{1i}\,dr\,d\theta_i+\sum_{ij}a_{ij}\,d\theta_i\,d\theta_j$

exp : rays → geodesics, preserving length ⇒ $a_{11}=1$. Gauss lemma ⇒ $a_{1i}=0$.

∴ $\frac{\partial}{\partial r}$ = the gradient of $r=(x_1^2+\cdots+x_n^2)^{\frac12}$. $x_1,\cdots,x_n$: normal coordinates

Gauss lemma ⇒ (Geodesics ⇒ locally shortest curves).

Corollary Let $B_r(0)\subseteq T_pM$ be a ball of radius r on which $\exp_p$ is an embedding. Then for $v\in B_r(0)$, $\gamma_v(t):[0,1]\to M$ is the unique curve satisfying

$L[\gamma]=d(p,\exp_p(v))=\|v\|$. In particular, for any curve c, if $L[c]=d(c(0),c(1))$, then, up to reparametrization, c is a smooth geodesic.

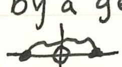
pf) Let $c:[0,1]\to M$ be a C^∞ curve from p to $\exp_p(v)$. Assume $c\subset\exp_p(B_r(0))$.

Since $\|\frac{\partial}{\partial r}\|=\|\nabla r\|\equiv1$, we have $\|c'(t)\|\geq\langle c'(t),\frac{\partial}{\partial r}\rangle=\frac{d}{dt}r(c(t))$.

∴ $L[c]=\int_0^1\|c'\|dt\geq\int_0^1\langle c',\frac{\partial}{\partial r}\rangle dt=\int_0^1\frac{d}{dt}r(c(t))=r(c(t))\Big|_0^1=r(c(1))=\|v\|$.

∴ $L[c]=\|v\|$ ⇔ $c'(t)=\lambda(t)\frac{\partial}{\partial r}$, $\lambda(t)\geq0$ ⇔ $c(t)$: a radial geodesic up to reparametrization.

When is it possible to join two arbitrary points by a geodesic whose length is equal to the distance between them?



When is $\exp_p$ defined on all of T_pM?