proof) $X\langle Y,Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$

$Y\langle Z,X\rangle = \langle \nabla_Y Z, X\rangle + \langle Z, \nabla_Y X\rangle$

$Z\langle X,Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle.$

$\therefore X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle = \langle [X,Z],Y\rangle + \langle [Y,Z],X\rangle + \langle [X,Y],Z\rangle + 2\langle Z,\nabla_Y X\rangle$

Therefore $\langle Z,\nabla_Y X\rangle = \frac{1}{2}\{X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle$
$- \langle [X,Z],Y\rangle - \langle [Y,Z],X\rangle - \langle [X,Y],Z\rangle\}$. $\Rightarrow$ Uniqueness

∇ is well defined by $\langle Z,\nabla_Y X\rangle = \cdots$.

Def Let $\sigma : [0,1] \to M$ be a C^∞ curve. For any $v \in T_{\sigma(0)}M$ there exists a unique vector field $V(t)$ along σ such that $V(t) \in T_{\sigma(t)}M$, $V(0) = v$, and $\nabla_{\sigma'(t)} V(t) \equiv 0$. We call $V(t)$ a parallel field and $V(1)$ the parallel translate of v along $\sigma(t)$.

proof) One can find C^∞ vector fields along $\sigma(t)$, $E_1(t), \cdots, E_n(t)$ s.t. $\{E_i(t)\}$ is an orthonormal basis for $T_{\sigma(t)}M$.

$\frac{d}{dt}\langle V,E_i\rangle = \sigma'(t)\langle V,E_i\rangle = \langle \nabla_{\sigma'} V, E_i\rangle + \langle V, \nabla_{\sigma'} E_i\rangle = \langle \nabla_{\sigma'} E_i, \sum_j \langle V,E_j\rangle E_j\rangle$

$$\therefore \quad \begin{bmatrix} \langle V,E_1\rangle \\ \vdots \\ \langle V,E_n\rangle \end{bmatrix}' = \langle \nabla_{\sigma'} E_i, E_j\rangle \begin{bmatrix} \langle V,E_1\rangle \\ \vdots \\ \langle V,E_n\rangle \end{bmatrix}$$
: a system of ODE's (linear) $\therefore$ existence & uniqueness

V,W : parallel $\Rightarrow \sigma'\langle V,W\rangle = \langle \nabla_{\sigma'} V, W\rangle + \langle V, \nabla_{\sigma'} W\rangle = 0$. $\therefore P_\sigma$ is an isometry.

$P_\sigma : T_{\sigma(0)}M \to T_{\sigma(1)}M$, $P_\sigma(v) = V(1)$: parallel translation : linear $\because$ linear ODE.

connection $\nabla \leftrightarrow$ parallel vector fields $W_1, \cdots, W_n$, $W = \sum f_i(t) W_i$

$\nabla_v W = \sum v(f_i) W_i$

Def A smooth curve c is a geodesic if $\nabla_{c'} c' \equiv 0$.

c: geodesic $\Rightarrow c'\langle c',c'\rangle = 2\langle \nabla_{c'} c', c'\rangle = 0 \Rightarrow$ constant speed

$\Rightarrow$ A geodesic is parametrized proportional to arclength.

line in $\mathbb{R}^n$: straight, shortest. geodesic in M : straight $\because \nabla_{c'} c' = 0$

$x_1, \cdots, x_n$: coordinates along $c(t)$. $\underset{dc(\frac{d}{dt})}{\overset{\shortparallel}{c'}} = \frac{dx_1}{dt}\frac{\partial}{\partial x_1} + \cdots + \frac{dx_n}{dt}\frac{\partial}{\partial x_n}$

$$\nabla_{c'} c' = \nabla_{c'}\left(\frac{dx_1}{dt}\frac{\partial}{\partial x_1}+\cdots+\frac{dx_n}{dt}\frac{\partial}{\partial x_n}\right) = \frac{d^2x_1}{dt^2}\frac{\partial}{\partial x_1}+\cdots+\frac{d^2x_n}{dt^2}\frac{\partial}{\partial x_n}+\sum_{ij}\frac{dx_i}{dt}\nabla_{\frac{\partial}{\partial x_i}}\left(\frac{dx_j}{dt}\frac{\partial}{\partial x_j}\right)$$

$$\therefore \frac{d^2x_k}{dt^2}+\sum_{ij}\Gamma_{ij}^k\frac{dx_i}{dt}\frac{dx_j}{dt}=0,\quad k=1,\cdots,n \quad (*) \qquad \sum_k \Gamma_{ij}^k\frac{\partial}{\partial x_k}=\nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_j}. \text{ Christoffel.}$$

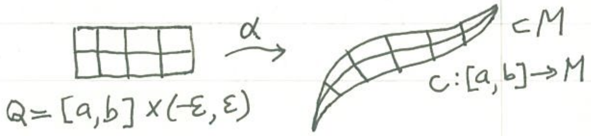
$$\Gamma_{ij}^k=\frac{1}{2}\sum_m\left\{\frac{\partial}{\partial x_i}g_{jm}+\frac{\partial}{\partial x_j}g_{mi}-\frac{\partial}{\partial x_m}g_{ij}\right\}g^{mk}.$$

By the existence and uniqueness for the 2nd order ODE, there exists a unique geodesic γ_v through p with tangent v for any $p\in M$ and $v\in T_pM$.

<u>Def</u> The <u>exponential</u> <u>map</u> $\exp_p: T_pM \to M$ is defined by $\exp_p(v)=\gamma_v(1)$ for all $v\in T_pM$ such that 1 is in the domain of γ_v.
If $\gamma_v:(-\varepsilon,\varepsilon)\to M$ is a geodesic, the curve $c:(\frac{\varepsilon}{s},\frac{\varepsilon}{s})\to M$ defined by $c(t)=\gamma_v(st)$ is also a geodesic. $\therefore$ for s small, 1 is in the domain of c.
$\exp_p$ is defined in a neighborhood of the origin in T_pM, and is a local diffeomorphism because $d\exp_p = $ identity.

$\{e_1,\cdots,e_n\}$: orthonormal basis for T_pM, $x_1,\cdots,x_n$: coordinates on T_pM.
Assign to the point $\exp_p(\sum x_ie_i)$ the coordinates $(x_1,\cdots,x_n)$: <u>normal</u> <u>coordinates</u> in a nbhd of p.
The rays through p are geodesics. $\Rightarrow \nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_i}\equiv 0 \Rightarrow \left(\nabla_v\frac{\partial}{\partial x_i}\right)\Big|_p=0, \forall v\in T_pM$.
$\therefore$ At p $x_1,\cdots,x_n$ are like Euclidean coordinates: convenient.

<u>First Variation of Arc Length</u>



$\alpha: Q\to M$ smooth map, $\alpha|_{[a,b]\times\{0\}}=c$
α is said to define a <u>smooth variation</u> of the curve $\alpha|([a,b],0)$.
$t\in[a,b]$, $s\in(-\varepsilon,\varepsilon)$. $T=\frac{\partial}{\partial t}$, $V=\frac{\partial}{\partial s}$: We shall identify these vectors with their images under $d\alpha$.
Goal: Compute the change in arc length over the family of curves $c_s=\alpha|_{[a,b]\times\{s\}}$ where $-\varepsilon<s<\varepsilon$.
Assume c to be parametrized proportional to arc length, i.e., $\|c'(t)\|=l$ constant

$$\frac{d}{ds}L[c_s]=\frac{d}{ds}\int_a^b\langle c_s'(t),c_s'(t)\rangle^{\frac{1}{2}}dt = \int_a^b V\langle T,T\rangle^{\frac{1}{2}}dt$$