can be immersed in $\mathbb{R}^{2n-1}$ and embedded in $\mathbb{R}^{2n}$. If M is compact, orientable then M embeds in $\mathbb{R}^{2n-1}$.

<u>Def</u> A family $\{f_\alpha\}$ of C^∞ functions $f_\alpha : M \to \mathbb{R}$ is a C^∞ <u>partition of unity</u> if (a) For all α, $f_\alpha \geq 0$ and the support of f_α is contained in a coordinate nbhd U_α of a C^∞ structure $\{U_\alpha, \varphi_\alpha\}$ of M.
(b) The family $\{U_\alpha\}$ is locally finite.
(c) $\sum_\alpha f_\alpha \equiv 1$.

<u>Theorem</u> A C^∞ manifold M has a C^∞ partition of unity if and only if M is Hausdorff and has a countable basis.

RIEMANNIAN METRICS

<u>Definition</u> A <u>Riemannian metric</u> on a C^∞ manifold M is a C^∞ symmetric bilinear, positive definite form $\langle\ ,\ \rangle_p$ on the tangent space T_pM for each point p of M.

$x_1, \cdots, x_n$: coordinates on $U \subset M$. Then $\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \rangle = g_{ij}(x_1, \cdots, x_n)$ is a C^∞ function on U.
Equivalently, if $X, Y \in \mathfrak{X}(U)$, then $\langle X, Y \rangle \in C^\infty(U)$.
g_{ij} is called the local representation of the Riemannian metric in the coordinates $x_1, \cdots, x_n$.

A <u>Riemannian manifold</u> is a C^∞ manifold with a given Riemannian metric.

<u>Def</u> M, N: Riemannian manifolds. A diffeomorphism $\varphi : M \to N$ is called an <u>isometry</u> if $\langle u, v \rangle_p = \langle d\varphi_p(u), d\varphi_p(v) \rangle_{\varphi(p)}$, $u, v \in T_pM$ for all $p \in M$.
<u>local isometry</u>

ex) $\mathbb{R}^n$, $g_{ij} = \delta_{ij}$. $\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \rangle = \delta_{ij}$ Euclidean space

ex) $\varphi : M^n \to N^{n+k}$ immersion, N: Riemannian manifold. Then φ induces a Riemannian metric on M by $\langle u, v \rangle_p = \langle d\varphi_p(u), d\varphi_p(v) \rangle_{\varphi(p)}$, $u, v \in T_p(M)$.
$d\varphi_p$: injective $\Rightarrow$ $\langle\ ,\ \rangle_p$ is positive definite. φ: isometric immersion.
$M \subset N$, submanifold : induced metric

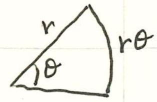
* Nash embedding theorem: A Riemannian manifold M^n can be isometrically embedded in $\mathbb{R}^m$ with $m \le n(3n+11)/2$ if M compact, $m \le n(n+1)(3n+11)/2$ if noncompact.

Any two 1-dimensional Riemannian manifolds are locally isometric. 10

ex) $i: S^n \to \mathbb{R}^{n+1}$ inclusion. The induced metric is called the canonical metric of S^n.

* $x_1, \cdots, x_n$, $g_{ij} = \langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \rangle$. $g = \sum_{i,j} g_{ij}\, dx_i\, dx_j$

$\mathbb{R}^n$: $g = dx_1^2 + \cdots + dx_n^2$



$\mathbb{R}^2$: $g = dx^2 + dy^2 = dr^2 + r^2 d\theta^2$: r, θ : polar coordinates

S^2: r, θ $g = dr^2 + \sin^2 r\, d\theta^2$

H^2: r, θ $g = dr^2 + \sinh^2 r\, d\theta^2$

S^2: x, y : by the stereographic projection $g = \dfrac{dx^2 + dy^2}{\left(1 + \frac{x^2+y^2}{4}\right)^2}$

ex) $\mathbb{R}^2$: $x = r\cos\theta$, $y = r\sin\theta$

$\frac{\partial}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} = \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y}$, $\frac{\partial}{\partial \theta} = \frac{\partial x}{\partial \theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial}{\partial y} = -r\sin\theta \frac{\partial}{\partial x} + r\cos\theta \frac{\partial}{\partial \theta}$

$\therefore \underbrace{\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial r} \rangle}_{g_{11}} = 1$, $\underbrace{\langle \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \theta} \rangle}_{g_{22}} = r^2$, $\underbrace{\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta} \rangle}_{g_{12}} = 0$ $\therefore g = dr^2 + r^2 d\theta^2$

* Let (M, g) and (N, h) be two Riemannian manifolds. The <u>product metric</u> $g \times h$ on $M \times N$ is defined by

$(g\times h)_{(p,q)}(u, v) = g_p(d\pi_1(u), d\pi_1(v)) + h_q(d\pi_2(u), d\pi_2(v))$,

$\pi_1: M \times N \to M$, $\pi_2: M \times N \to N$: natural projections.

ex) $T^n = S^1 \times \cdots \times S^1$, $S^1 \subset \mathbb{R}^2$ induced metric "flat torus" rectangular torus?

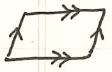
ex) $g_1: S^2$, $dr^2: (0, \infty)$: canonical metrics

$g = r^2 g_1 + dr^2$: flat metric on $S^2 \times (0,\infty) \simeq \mathbb{R}^3 \setminus \{0\}$

$g_2: S^1$, $g = g_2 + dr^2 \Rightarrow$ $(S^1 \times \mathbb{R}^1, g)$ is isometric to the cylinder $C = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$ with the metric induced from $\mathbb{R}^3$.

ex) $p: S^2 \subset \mathbb{R}^3 \to P^2$, covering map: local diffeomorphism

$\therefore$ push-forward metric on P^2.

ex)  torus: use covering map. not a product metric

* $\sigma: [a,b] \to M$, curve $\dot\sigma(t) = d\sigma(\frac{d}{dt})$: velocity of σ : vector field along σ

$\ell(\sigma) = \int_a^b \sqrt{g(\dot\sigma, \dot\sigma)}\, dt$: the <u>length</u> of σ.

* $U, V \subset M$, (U,φ), (V,ψ) coordinate systems with $x_1,\cdots,x_n, y_1,\cdots,y_n$.
$\{e_1,\cdots,e_n\}$ orthonormal basis of T_pM, $\forall p\in U$. $\frac{\partial}{\partial x_i} = \sum_j a_{ij} e_j$. Then
$g_{ik} = \langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_k} \rangle = \sum_{j\ell} a_{ij} a_{k\ell} \langle e_j, e_\ell \rangle = \sum_j a_{ij} a_{kj}$
$\therefore G := (g_{ij})$, $A =: (a_{ij})$. Then $G = AA^T$, $\det G = \det(AA^T) = (\det A)^2$.
volume of the parallelepiped formed by $\frac{\partial}{\partial x_1},\cdots,\frac{\partial}{\partial x_n} = \det(\frac{\partial}{\partial x_1},\cdots,\frac{\partial}{\partial x_n}) = \det A$.
$R \subset U\cap V$. Define the <u>volume</u> of R by

$$vol(R) = \int_{\varphi(R)} \sqrt{\det(g_{ij})}\, dx_1 \cdots dx_n .$$

$\frac{\partial}{\partial x_i} = \sum_j \frac{\partial y_j}{\partial x_i}\frac{\partial}{\partial y_j}$ $\therefore$ $vol(\frac{\partial}{\partial x_1},\cdots,\frac{\partial}{\partial x_n}) = \det(\frac{\partial y_j}{\partial x_i})\, vol(\frac{\partial}{\partial y_1},\cdots,\frac{\partial}{\partial y_n})$
$= \det(\frac{\partial y_j}{\partial x_i}) \sqrt{\det(h_{ij})}$, $h_{ij} = \langle \frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j} \rangle$.
$\therefore vol(R) = \int_{\varphi(R)} \sqrt{\det(g_{ij})}\, dx_1\cdots dx_n = \int_{\psi(R)} \sqrt{\det(h_{ij})}\, dy_1 \cdots dy_n$.
Therefore the volume is well defined.

ex) S^3: $S^3 \setminus \{\text{north pole, south pole}\} = (0,\pi)\times S^2$ with metric $g = dr^2 + \sin^2 r\, h$,
dr^2 and h are the standard metrics on $(0,\pi)$ and S^2, respectively.
$vol(S^3) = \int_0^\pi vol(\{r=t\})\,dt = \int_0^\pi 4\pi \sin^2 t\, dt = 2\pi^2$

CONNECTIONS

<u>Def</u> An <u>affine connection</u> is a bilinear map $\nabla: \mathfrak{X}(M)\times\mathfrak{X}(M)\to\mathfrak{X}(M)$
denoted by $(X,Y) \xrightarrow{\nabla} \nabla_X Y$ which satisfies the following properties:
(a) $\nabla_{fX} Y = f\nabla_X Y$ (b) $\nabla_X fY = X(f)Y + f\nabla_X Y$ for any $f\in C^\infty(M)$, $X, Y \in \mathfrak{X}(M)$.
We call $\nabla_X Y$ the <u>covariant</u> <u>derivative</u> of Y in the direction of X.
* $(\nabla_X Y)(p)$ is determined by $X(p)$ and Y restricted to any curve through p in direction $X(p)$.

<u>Fundamental Theorem of Riemannian Geometry</u>

Given a Riemannian manifold M, there exists a unique affine connection ∇ on M with the following two properties:

(a) $\nabla_X Y - \nabla_Y X = [X,Y]$ (b) $X\langle Y,Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$, $\forall X,Y,Z\in\mathfrak{X}(M)$
This connection will be referred to as the <u>Levi-Civita</u> (or <u>Riemannian</u>) <u>connection</u> on M.