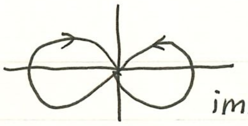
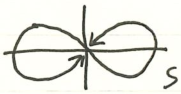
ex) $\psi : \mathbb{R}^1 \to \mathbb{R}^2$, $\psi(t) = (2\cos t, \sin 2t)$



immersion, but not a submfd

$t = 2\tan^{-1}\theta - \frac{\pi}{2}$, $\theta \mapsto \psi(t)$



submanifold but not an embedding

$\psi : \mathbb{R}^1 \to \mathbb{R}^2$, $\psi(t) = (t, \sin t)$

embedding

* The distinction between immersions and embeddings is a global one.
ψ : immersion $\Rightarrow$ each $p \in M$ has a nbhd U s.t. $\psi|_U$ is an embedding.

* $\psi : \mathbb{R}^n \to \mathbb{R}^n$ diffeomorphism $\Rightarrow d\psi_p$: isomorphism. (converse? $\Rightarrow$ IFT)

<u>Inverse Function Theorem</u>

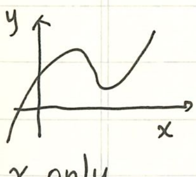
$U \subset \mathbb{R}^n$ open, $f : U \to \mathbb{R}^n$ C^∞. If the Jacobian matrix $\left\{\frac{\partial r_i \circ f}{\partial r_j}\right\}$ is nonsingular at $r_0 \in U$, then there exists an open set V with $r_0 \in V \subset U$ s.t. $f|_V$ maps V 1-1 onto the open set $f(V)$, and $(f|_V)^{-1}$ is C^∞.

<u>Corollary</u> Suppose that $\dim M = n$ and that $y_1, \cdots, y_n$ are C^∞ functions in a nbhd U of $p \in M$. ~~such that $\frac{\partial}{\partial y_1}, \cdots, \frac{\partial}{\partial y_n}$ are linearly independent vectors in T_pM~~. Then the functions $y_1, \cdots, y_n$ form a coordinate system in a nbhd of p. — If there exist vectors $v_1, \cdots, v_n \in T_pM$ s.t. $(v_i(y_j))$ is nonsingular,

pf) Define $\psi : U \to \mathbb{R}^n$, $\psi(m) = (y_1(m), \cdots, y_n(m))$. Then $d\psi_p$ is an isomorphism.

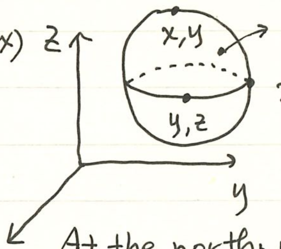
<u>Corollary</u> Let $\psi : M \to N$ be C^∞ and assume that $d\psi : T_pM \to T_{\psi(p)}N$ is injective. Let $x_1, \cdots, x_k$ form a coordinate system on a nbhd of $\psi(p)$. Then a subset of the functions $\{x_i \circ \psi\}$ forms a coordinate system on a nbhd of p. In particular, ψ is 1-1 on a nbhd of p.

ex)



x only.

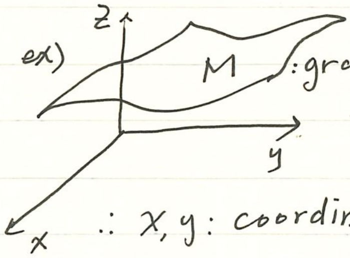
ex)



any two of x, y, z !

At the north pole p
$v(z) = 0$, $\forall v \in T_pM$.

ex)



: graph of $z = f(x,y)$
$f \in C^\infty(\mathbb{R}^2)$.

$\therefore$ x, y : coordinates on M.

ex) $i : \mathbb{R}^2 \to \mathbb{R}^3$

<u>Corollary</u> Let $\psi : M \to N$ be C^∞, and assume that $d\psi : T_pM \to T_{\psi(p)}N$ is surjective. Let $x_1, \cdots, x_\ell$ form a coordinate system on some nbhd of $\psi(p)$. Then $x_1 \circ \psi, \cdots, x_\ell \circ \psi$ form part of a coordinate system on some nbhd of p. ex) $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ projection

<u>Theorem</u> Assume that $\psi : M^c \to N^d$ is C^∞, that q is a point of N, that $O = \psi^{-1}(q)$ is nonempty, and that $d\psi : T_pM \to T_{\psi(p)}$ is surjective for all $p \in O$. Then O has a unique manifold structure s.t. (O, i) is a submanifold of M, where i is the inclusion map. Moreover, $i : O \to M$ is actually an embedding, and the dimension of O is $c-d$.
(sketchy proof) $x_1, \cdots, x_d$: coordinate system centered at $q \in N$.
$z_1, \cdots, z_c$: coordinate system on a nbhd U of $p \in O$. $d\psi$: surjective forall $p \in O \Rightarrow \{y_i = x_i \circ \psi : i = 1, \cdots, d\}$, $\{y_1, \cdots, y_d, z_{d+1}, \cdots, z_c\}$: coordinate system on a nbhd U of p. $\Rightarrow O \cap U = \{y_1 = y_2 = \cdots = y_d = 0\}$: the slice of this coordinate system which is a submanifold with a coordinate system $z_{d+1}, \cdots, z_c$ $\therefore$ of dimension $c-d$.

ex) $N_1 = \{z=0\}$, $N_2 = \{z = x^2 - y^2\}$, $M = N_1 \cap N_2$. M is not a submanifold
$\psi : N_2 \to \mathbb{R}^1$, $\psi(x,y,z) = z = x^2 - y^2$. $\psi : M \to N$
$$d\psi\left(\frac{\partial}{\partial x_j}\Big|_p\right) = \sum \frac{\partial(y_i \circ \psi)}{\partial x_j}\Big|_p \frac{\partial}{\partial y_i}\Big|_{\psi(p)}, \quad (U, x_1, \cdots, x_d) \text{ on } M,\ (V, y_1, \cdots, y_\ell) \text{ on } N$$
Jacobian of $\psi = (2x, -2y)$ for the coordinate system x, y on N_2.
$\therefore d\psi$ is not surjective at $(0,0) \in \psi^{-1}(\psi(0,0))$.
ex) $f(p) = \sum_{i=1}^{d} r_i^2$ on $\mathbb{R}^d$. df : surjective except at the origin. $\therefore f^{-1}(r^2)$: submfd if $r>0$.
ex) $Gl(n,\mathbb{R}) \supset O(n,\mathbb{R})$, $\psi : Gl(n,\mathbb{R}) \to Gl(n,\mathbb{R})$, $\psi(A) = AA^t$.
$\mathrm{Im}\,\psi \subset \{\text{symmetric matrices}\} := S(n)$, $O(n,\mathbb{R}) = \psi^{-1}(I)$.
$d\psi : T_A Gl(n,\mathbb{R}) \to T_{\psi(A)} Gl(n,\mathbb{R}) \supset S(n) = \mathbb{R}^{\frac{1}{2}(n+1)n}$
$\quad = M(n) = \mathbb{R}^{n^2}$ $\qquad d\psi_A(B) = ?$ $\quad \sigma(t) = A + tB$: curve with $\sigma'(0) = B$.
$$\therefore d\psi_A(B) = \frac{d}{dt}\psi(A+tB)\Big|_{t=0} = \lim_{t\to 0} \frac{1}{t}\left[(A+tB)(A+tB)^t - AA^t\right]$$
$$= \lim_{t\to 0} \frac{1}{t}\left[BA^t + AB^t + tBB^t\right] = BA^t + AB^t.$$

$\forall C \in S(n)$, $C = \frac{1}{2}C + \frac{1}{2}C^t$. If $B = \frac{CA}{2}$, then $BA^t = \frac{1}{2}C$,

$d\psi_A(B) = BA^t + AB^t = \frac{1}{2}C + \frac{1}{2}C^t = C$ $\therefore$ $d\psi_A$ is surjective on $S(n)$, $\forall A \in \psi^{-1}(I)$.

$\therefore$ $O(n, \mathbb{R})$ is a submanifold of dimension $\frac{1}{2}n(n-1)$.

* Vector Fields

Def A C^∞ vector field X on an open set U in M is a C^∞ lifting of U into $T(M)$, that is, a C^∞ map $X : U \to T(M)$ s.t. $\pi \circ X =$ identity map on U.

$\therefore$ $X(p) \in T_pM$. $f \in C^\infty(U) \Rightarrow fX$: vector field.

$\mathfrak{X}(U) := \{C^\infty \text{ vector fields on } U\}$: a vector space over $\mathbb{R}$

Proposition X : vector field on M. The following are equivalent:

(a) X is C^∞.

(b) If $(U, x_1, \cdots, x_d)$ is a coordinate system on M and $X|_U = \sum_{i=1}^{d} a_i \frac{\partial}{\partial x_i}$, then $a_i \in C^\infty(U)$.

(c) If $f \in C^\infty(U)$, then $X(f) \in C^\infty(U)$.

Lemma X, Y : C^∞ vector fields on M. Then there exists a unique C^∞ vector field Z s.t. for all $f \in C^\infty(M)$, $Zf = (XY - YX)f$.

$Z := [X, Y]$: the Lie bracket of X and Y.

Proposition (a) If $f, g \in C^\infty(M)$, then $[fX, gY] = fg[X,Y] + f(Xg)Y - g(Yf)X$.

(b) $[X,Y] = -[Y,X]$ (c) $[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$.

(c) : Jacobi identity (c), (d) : Lie algebra

Curve $\sigma \to \dot{\sigma}(t)$. vector field $\to$ curve ?

Def $X \in \mathfrak{X}(M)$. A smooth curve σ in M is an integral curve of X if $\dot{\sigma}(t) = X(\sigma(t))$, $\forall t$.

ex) $\nabla(x^2+y^2)$

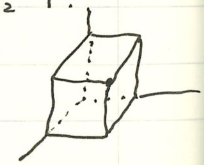
Does X have an integral curve? Is it unique?

$\sigma : (a,b) \to M$ is an integral curve of $X \Leftrightarrow d\sigma(\frac{d}{dr}|_t) = X(\sigma(t))$, $t \in (a,b)$.
$0 \in (a,b)$, $\sigma(0) = p$, $x_1, \cdots, x_d$: coordinates about p in U.
$X|_U = \sum f_i \frac{\partial}{\partial x_i}$, $f_i \in C^\infty(U)$. $d\sigma(\frac{d}{dr}|_t) = \sum \frac{d(x_i \circ \sigma)}{dr}|_t \frac{\partial}{\partial x_i}|_{\sigma(t)}$
$= \sum f_i(\sigma(t)) \frac{\partial}{\partial x_i}|_{\sigma(t)}$
$\therefore$ σ is an integral curve of X on $\sigma^{-1}(U)$ if and only if
$\frac{d\sigma_i}{dr}|_t = f_i(\sigma_1(t), \cdots, \sigma_d(t))$, $\sigma_i = x_i \circ \sigma$, $i = 1, \cdots, d$, $t \in \sigma^{-1}(U)$.
: A sytem of first order ODE's. Existence & Uniqueness.

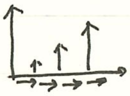
Def Define a transformation X_t by $X_t(p) = \sigma_p(t)$, for each $t \in \mathbb{R}$.
Theorem X_t is a diffeomorphism with inverse X_{-t}. $X_s \circ X_t = X_{s+t}$.
ex) $y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$, $x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$, ∇r^2, ∇r^3

Proposition $p \in M^d$, $X \in \mathfrak{X}(M)$, $X(p) \neq 0$. Then there exists a coordinate system (U, φ) with coordinate functions $x_1, \cdots, x_d$ on a nbhd of p s.t. $X|_U = \frac{\partial}{\partial x_1}|_U$.

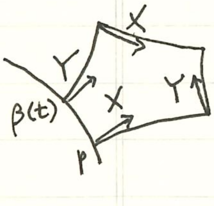
$X, Y \in \mathfrak{X}(M)$. Does there exist coordinates $x_1, \cdots, x_d$ s.t. $\frac{\partial}{\partial x_1} = X$, $\frac{\partial}{\partial x_2} = Y$?
Theorem $X, Y \in \mathfrak{X}(M)$. $[X,Y] = 0 \Leftrightarrow X_t \circ Y_s = Y_s \circ X_t$.
$\therefore$ $[X,Y] = 0 \Leftrightarrow \exists$ coord. system $x_1, \cdots, x_d$ s.t. $\frac{\partial}{\partial x_1} = X$, $\frac{\partial}{\partial x_2} = Y$.
$[X_i, X_j] = 0 \Leftrightarrow \exists$ coordinates $x_1, \cdots, x_d$ s.t. $X_i = \frac{\partial}{\partial x_i}$



ex) $\frac{\partial}{\partial x}$, $x\frac{\partial}{\partial y}$



* $[X,Y]_p f = \lim_{t \to 0} \frac{f(\beta(t)) - f(\beta(0))}{t}$, $\beta(t) = Y_{-\sqrt{t}} X_{-\sqrt{t}} Y_{\sqrt{t}} X_{\sqrt{t}}(p)$



$[X,Y]$ measures how different $X_t \circ Y_s$ is from $Y_s \circ X_t$.

*Whitney embedding theorem. Any C^∞ manifold M of dimension n