$\varphi^{-1*}(d\omega) = d\varphi^{-1*}(\omega) = \Sigma \frac{\partial \lambda_j}{\partial r_j} dr_1 \wedge \cdots \wedge dr_n$.

$\therefore \int_M d\omega = \sum_j \int_0^1 \cdots \int_0^1 \frac{\partial \lambda_j}{\partial r_j} dr_1 \cdots dr_n = \int_0^1 \cdots \int_0^1 [\lambda_j(r_1, \ldots, r_{j-1}, 1, r_{j+1}, \cdots, r_n)$
$- \lambda_j(r_1, \ldots, r_{j-1}, 0, r_{j+1}, \cdots, r_n)] dr_1 \cdots \widehat{dr_j} \cdots dr_n$

$= 0$ if $\text{supp}(\omega) \cap \partial M = \phi$, and $\overset{\text{otherwise}}{=} - \int_0^1 \cdots \int_0^1 \lambda_n(r_1, \cdots, r_{n-1}, 0) dr_1 \cdots dr_{n-1}$.

$\varphi^{-1*}(\omega|_{\partial M}) = (-1)^{n-1} \lambda_n(r_1, \ldots, r_{n-1}, 0) dr_1 \wedge \cdots \wedge dr_{n-1}$.

$\therefore \int_{\partial M} \omega = (-1)^{n-1} \int_0^1 \cdots \int_0^1 \lambda_n(r_1, \cdots, r_{n-1}, 0) dr_1 \cdots dr_{n-1}$

$\therefore \int_M d\omega = (-1)^n \int_{\partial M} \omega = \int_{\partial M} \omega$ after change of orientation in such a way that $v_1, \cdots, v_{n-1}$ is an oriented basis of $T_p\partial M \overset{\text{def}}{\Longleftrightarrow} v_1, \cdots, v_{n-1}, \eta$ is an oriented basis of T_pM, where η is an outer vector to ∂M at p.

* Integration on a Riemannian manifold

Metric $\langle \, , \rangle$ gives a canonical isomorphism $\mathfrak{X}(M) \cong E^1(M)$. $X \in \mathfrak{X}(M) \Rightarrow \tilde{X} \in E^1(M)$, $\tilde{X}(Y) = \langle X, Y \rangle$. $\omega \in E^1(M) \Rightarrow \tilde{\omega} \in \mathfrak{X}(M)$, $\langle \tilde{\omega}, Y \rangle = \omega(Y)$. $\tilde{X} = X^\flat, \tilde{\omega} = \omega^\sharp$.

$x_1, \cdots, x_n$: local coordinates on $M \Rightarrow \frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n} \in \mathfrak{X}(M) \xRightarrow{\text{Gram-Schmidt}}$ Orthonormal vector fields $e_1, \cdots, e_n \in \mathfrak{X}(M) \Rightarrow \tilde{e}_1 = \omega_1, \cdots, \tilde{e}_n = \omega_n \in E^1(M)$. Suppose $\omega_1 \wedge \cdots \wedge \omega_n$ determines the orientation of M. $\omega_1 \wedge \cdots \wedge \omega_n$ agrees on overlaps of coordinate domains. $\therefore \omega_1 \wedge \cdots \wedge \omega_n$ is a globally defined nowhere-vanishing n-form ω on M. ω is called the <u>volume form</u> of the oriented Riemannian manifold M. Its integral over M is the <u>volume</u> of M.

$g_{ij} = \langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \rangle = \langle \Sigma \alpha_i^k X_k, \Sigma \alpha_j^l X_l \rangle = \sum_k \alpha_i^k \alpha_j^k$. $\left(\frac{\partial}{\partial x_i} = \Sigma \alpha_i^k X_k \right)$

$\therefore (g_{ij}) = A^t A$, $A = (\alpha_i^k)$. $\omega(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}) = \omega(\Sigma \alpha_1^k X_k, \cdots, \Sigma \alpha_n^k X_k)$

$= \det(\alpha_i^k) = \sqrt{\det(g_{ij})}\, dx_1 \wedge \cdots \wedge dx_n (\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})$. $\therefore \omega = \sqrt{g}\, dx_1 \wedge \cdots \wedge dx_n$,

$g := \det(g_{ij})$.

linear operator $* : E^p(M) \to E^{n-p}(M)$ on an oriented Riemannian manifold M^n

$*(1) = \omega_1 \wedge \cdots \wedge \omega_n$, $*(\omega_1 \wedge \cdots \wedge \omega_n) = 1$, $*(\omega_1 \wedge \cdots \wedge \omega_p) = \omega_{p+1} \wedge \cdots \wedge \omega_n$

$** = (-1)^{p(n-p)}$.

Define the integral over M of a continuous function f with compact support by $\int_M f = \int_M *f = \int_M f\omega$.

$\nabla f = \widetilde{df} = df^\sharp$, $\text{div}\, V = *d*\tilde{V}$, $V \in \mathfrak{X}(M)$.
$f \in C^\infty(M)$

Stokes' theorem $\Rightarrow$ the divergence theorem

If V is a smooth vector field with compact support on a Riemannian manifold M, if D is a domain with smooth boundary in M, and if n is the unit outer normal vector field on ∂D, then $\int_D \mathrm{div}\, V = \int_{\partial D} \langle V, n \rangle$.

$e_1, \cdots, e_{n-1}, -n$: orthonormal vector fields along ∂D.

$V = \sum_{i=1}^{n-1} a_i e_i - a_n n$, $\tilde{V} = \sum_{i=1}^{n-1} a_i \omega_i + a_n \nu$, $* \tilde{V} = \sum a_i * \omega_i + a_n * \nu \overset{\text{on } \partial D}{=\!=} a_n * \nu$
$= (-1)^{n-1} a_n \omega_1 \wedge \cdots \wedge \omega_{n-1}$. $\omega_1 \wedge \cdots \wedge \omega_{n-1} \wedge \nu$: orientation of $M \Leftrightarrow$
$\omega_1 \wedge \cdots \wedge \omega_{n-1}$: orientation of ∂M. Stokes' $\Rightarrow \int_D \mathrm{div}\, V = \cdots = (-1)^n \int_{\partial D} * \tilde{V}$
$= (-1)^{2n-1} \int_{\partial D} a_n \omega_1 \wedge \cdots \wedge \omega_{n-1} = \int_{\partial D} \langle V, n \rangle$.

De Rham Cohomology

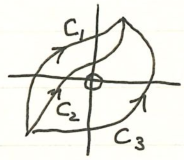
<u>Definition</u> A p-form α on M is called <u>closed</u> if $d\alpha = 0$. It is called <u>exact</u> if there is a $(p-1)$-form β such that $\alpha = d\beta$.

$d^2 = 0 \Rightarrow$ Every exact form is closed. Not every closed form is exact.

Poincaré Lemma : Any closed p-form on an open ball in $\mathbb{R}^n$ is exact.

ex) $f = \tan^{-1} \frac{y}{x}$ on $\mathbb{R}^2 \setminus \{0\}$: not well defined.

1-form $d\theta = \dfrac{-y\,dx + x\,dy}{x^2 + y^2} = d(\tan^{-1} \frac{y}{x})$ locally. θ is not a function



$d\theta$ is closed because $d\left(\dfrac{-y\,dx + x\,dy}{x^2 + y^2} \right) = 0$.

$\int_{C_1} d\theta = \int_{C_2} d\theta$, but $\int_{C_1} d\theta \neq \int_{C_3} d\theta$. $\int_{C_3 - C_1} d\theta = 2\pi$.

If α is an exact 1-form, then $\int_{C_1} \alpha = \int_{C_2} \alpha$ for any C_1, C_2 with same endpoints. $\because \int_{C_2 - C_1} \alpha = \int_{C_2 - C_1} d\beta = \int_{\underbrace{\partial(C_2 - C_1)}_{= \phi}} \beta = 0$.

$\therefore$ Exact forms cannot detect the topology of $\mathbb{R}^2 \setminus \{0\}$.

<u>Definition</u> The quotient space of the real vector space of closed p-forms modulo the subspace of exact p-forms is called the pth <u>de Rham cohomology group</u> of M. $H^p_{deR}(M) = \{\text{closed } p\text{-forms}\} / \{\text{exact } p\text{-forms}\}$.

ex) S^1. θ: polar coordinate function. not well defined (defined only up to integral multiples of 2π). However $d\theta$ is a globally well-defined nowhere vanishing 1-form on S^1. $d\theta$: volume form.

Not exact $\because \int_{S^1} d\theta \neq 0$. All 1-forms on S^1 are closed.

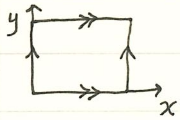
α: 1-form on $S^1 \Rightarrow \exists\, c$ s.t. $\alpha - c\,d\theta$ is exact. $\because \alpha := f(\theta)d\theta$, $c := \frac{1}{2\pi}\int_{S^1}\alpha$,

$g(\theta) := \int_0^\theta (f(\theta) - c)\,d\theta \Rightarrow g(\theta + 2n\pi) = g(\theta)$, $dg = (f(\theta) - c)d\theta = \alpha - c\,d\theta$.

$\therefore H^1_{deR}(S^1) \cong \mathbb{R}$. $\quad H^p_{deR}(S^1) = 0, \ \forall p \neq 0, 1$.

There are no exact 0-forms. A closed 0-form on a connected manifold is a constant function. $\therefore H^0_{deR}$(connected manifold) $\cong \mathbb{R}$.

ex) T^2 :



dx, dy: closed 1-forms $\quad H^1_{deR}(T^2) \cong \mathbb{R}\times\mathbb{R}$.

$dx \wedge dy$: closed 2-form. $\quad H^2_{deR}(T^2) \cong \mathbb{R}$.

* $f: M \to N$, C^∞ map. $\quad f^*: E^p(N) \to E^p(M)$ commutes with d.

$\therefore f^*$ maps closed forms to closed forms and exact forms to exact forms.

$\therefore f^*$ induces a homomorphism $H^p_{deR}(N) \to H^p_{deR}(M)$, $\forall p$.

$g: N \to P$, C^∞. Then $(g\circ f)^* = f^* \circ g^*$. $\quad id: M \to M$, $(id)^* = id$.

$\therefore$ A diffeomorphism $f: M \to N$ induces isomorphisms on de Rham cohomology.

ex) $\therefore S^2$ and T^2 are not diffeomorphic. $\because H^1_{deR}(S^2) \cong 0$, $H^2_{deR}(S^2) \cong \mathbb{R}$.

S^2: simply connected. $\Rightarrow$ Every closed 1-form on S^2 is exact.

ex) $V = f\mathbf{i} + g\mathbf{j} + h\mathbf{k}$ is a conservative vector field if $\int_{C_1} V\cdot d\vec{r} = \int_{C_2} V\cdot d\vec{r}$.

$k(x) := \int_p^x V\cdot d\vec{r} \Rightarrow dk = f\,dx + g\,dy + h\,dz \Leftrightarrow \nabla k = f\mathbf{i} + g\mathbf{j} + h\mathbf{k}$.

ex) $\alpha = f(x,y)\,dx + g(x,y)\,dy$, $d\alpha = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx \wedge dy$. $\therefore \alpha$ is closed if $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$. $\quad \alpha$ is exact if $\alpha = dh$, $\frac{\partial h}{\partial x} = f$, $\frac{\partial h}{\partial y} = g$. This α is closed because $\frac{\partial^2 h}{\partial x \partial y} = \frac{\partial^2 h}{\partial y \partial x}$.

ex) $\alpha = (2x + y\cos xy)\,dx + (x\cos xy)\,dy$ on $\mathbb{R}^2$. $\quad \alpha$ is exact.

<u>Definition</u> $\delta: E^p(M) \to E^{p-1}(M)$, $\quad \delta = (-1)^{n(p+1)+1} * d *$

Define the Laplace-Beltrami operator $\Delta: E^p(M) \to E^p(M)$ by $\Delta = \delta d + d\delta$.

$\Delta: E^0(\mathbb{R}^n) \to E^0(\mathbb{R}^n)$: $\quad \Delta = (-1)\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$.

<u>Definition</u> $H^p = \{\omega \in E^p(M) : \Delta\omega = 0\}$. $\quad$ harmonic p-forms.

<u>Theorem</u> (Hodge) Each de Rham cohomology class on a compact oriented Riemannian manifold M contains a unique harmonic representative.

ex) $\sigma = \dfrac{r_1\,dr_2 \wedge dr_3 - r_2\,dr_1 \wedge dr_3 + r_3\,dr_1 \wedge dr_2}{(r_1^2 + r_2^2 + r_3^2)^{\frac{3}{2}}}$: closed, volume form of S^2.

Theorem (Poincaré duality) $H^{n-p}_{deR}(M) \cong (H^p_{deR}(M))^*$.

Theorem (de Rham) $H^p_{deR}(M) \cong H_p(M;\mathbb{R})^*$

Define a bilinear function $H^{n-p}_{deR}(M) \times H^p_{deR}(M) \to \mathbb{R}$ by
$(\{\varphi\}, \{\psi\}) \longmapsto \int_M \varphi \wedge \psi$.

z: p-cycle in $H_p(M;\mathbb{R})$. Define $H^p_{deR}(M) \to H_p(M;\mathbb{R})^*$ by
$\{\alpha\}(\{z\}) = \int_z \alpha$.

Definition $\alpha, \beta \in E^p(M)$. $\langle \alpha, \beta \rangle := \dfrac{\alpha \wedge *\beta}{\omega} = \langle \beta, \alpha \rangle$

$\alpha = \sum_{i_1 < \cdots < i_p} \alpha_{i_1 \cdots i_p} \omega_{i_1} \wedge \cdots \wedge \omega_{i_p}$, $\beta = \sum_{j_1 < \cdots < j_p} \beta_{j_1 \cdots j_p} \omega_{j_1} \wedge \cdots \wedge \omega_{j_p}$

$\langle \alpha, \beta \rangle = \sum_{i_1 < \cdots < i_p} \alpha_{i_1 \cdots i_p} \beta_{i_1 \cdots i_p}$

Bochner's Formula

$$-\frac{1}{2}\Delta |\alpha|^2 = -\langle \Delta\alpha, \alpha \rangle + |\nabla \alpha|^2 + Ric(\alpha^\#, \alpha^\#).$$

Theorem (Bochner)

M^n: compact Riemannian manifold.

i) If $Ric > 0$, $b_1(M) = 0$ and $H^1_{deR}(M) = 0$.

ii) If $Ric \geqq 0$, then any harmonic 1-form on M is parallel and $b_1(M) \leq n$.

iii) If $Ric \geqq 0$ and $b_1(M) = n$, then $M =$ flat torus T^n.

α: harmonic 1-form.

pf) Bochner formula $\Rightarrow$ $|\alpha|^2$: superharmonic $\Rightarrow$ no interior maximum.
$\Rightarrow$ $|\alpha|^2$: const. $\Rightarrow$ α: parallel. $\therefore$ $Ric > 0 \Rightarrow \alpha \equiv 0$, $Ric \geqq 0 \Rightarrow$
$\dim H^1_{deR}(M) \leqq \dim T_pM = n$.