By definition, ψ^* is an algebra homomorphism; that is, $\psi^*(\omega\wedge\varphi)=\psi^*(\omega)\wedge\psi^*(\varphi)$.

$$d(\psi^*(\omega))=d(\sum a_{i_1\cdots i_k}\circ\psi\, d(x_{i_1}\circ\psi)\wedge\cdots\wedge d(x_{i_k}\circ\psi))=\sum(d(a_{i_1\cdots i_k}\circ\psi)\wedge d(x_{i_1}\circ\psi)\wedge\cdots\wedge d(x_{i_k}\circ\psi))$$
$$=\psi^*(\sum da_{i_1\cdots i_k}\wedge dx_{i_1}\wedge\cdots\wedge dx_{i_k})=\psi^*(d\omega).$$

Integration on Manifolds

V: a vector space, $\{e_1,\cdots,e_n\}$, $\{f_1,\cdots,f_n\}$: bases of V.
The bases are said to have the <u>same orientation</u> if $\det(\alpha_i^j)>0$ where $f_i=\sum_j \alpha_i^j e_j$, $i=1,\cdots,n$. This is an equivalence relation on the set of all bases of V, and there are exactly two equivalence classes.

<u>Definition</u> An <u>oriented vector space</u> is a vector space plus an equivalence class of allowable bases: The bases of a chosen equivalence class will be called <u>oriented</u> or <u>positively oriented</u> bases.

To extend the concept of orientation to a manifold M one must try to orient each of the tangent spaces T_pM in such a way that orientation of nearby tangent spaces agree.

<u>Definition</u> We shall call M <u>orientable</u> if there is a collection $\Phi=\{(V,\psi)\}$ of coordinate systems on M s.t. $M=\bigcup_{(V,\psi)\in\Phi}V$ and $\det\left(\frac{\partial x_i}{\partial y_j}\right)>0$ on $U\cap V$ whenever $(U,x_1,\cdots,x_n)$ and $(V,y_1,\cdots,y_n)$ belong to Φ.

Recall $\frac{\partial}{\partial y_j}=\sum\frac{\partial x_i}{\partial y_j}\frac{\partial}{\partial x_i}$. $\quad dy_j=\sum\frac{\partial y_j}{\partial x_i}dx_i$.

<u>Theorem</u> M is orientable if and only if there is a nowhere-vanishing n-form on M.

<u>Definition</u> A C^∞ <u>partition of unity</u> on M is a collection of C^∞ functions $\{f_i\}$ defined on M with the following properties:

(1) $f_i\geqq 0$ on M, $\qquad\qquad \mathrm{supp}(f)=\overline{f^{-1}(\mathbb{R}-\{0\})}$.
(2) $\{\mathrm{supp}(f_i)\}$ form a locally finite covering of M, and
(3) $\sum_i f_i(p)=1$ for every $p\in M$.

A collection $\{A_\alpha\}$ of subsets of M is <u>locally finite</u> if whenever $p\in M$ there exists a neighborhood W_p of p such that $W_p\cap A_\alpha\neq\phi$ for only finitely many α. A partition of unity $\{f_i : i\in I\}$ is <u>subordinate</u> to the cover

$\{U_\alpha : \alpha \in A\}$ if for each i there exists an α such that $\mathrm{supp}(f_i) \subset U_\alpha$.

Theorem (Existence of Partition of Unity) Let M be a differentiable manifold and $\{U_\alpha : \alpha \in A\}$ an open cover of M. Then there exists a countable partition of unity $\{f_i : i=1,2,3,\cdots\}$ subordinate to the cover $\{U_\alpha\}$ with $\mathrm{supp}(f_i)$ compact for each i.

pf) ($\Rightarrow$) $\{f_i\}$: partition of unity subordinate to the cover of M given by the coordinate neighborhoods in the collection Φ with f_i subordinate to $(V_i, x_1^i, \cdots, x_n^i)$. Then $\omega = \sum_i f_i\, dx_1^i \wedge \cdots \wedge dx_n^i$ is a global n-form on M, where $f_i\, dx_1^i \wedge \cdots \wedge dx_n^i$ is defined to be the 0 n-form outside of V_i. That ω vanishes nowhere follows from the fact that for each p, $\omega(p)$ is a finite sum with positive coefficients ($\because (dy_1 \wedge \cdots \wedge dy_n)(p) = \det\left(\frac{\partial y_j}{dx_i}\right)(dx_1 \wedge \cdots \wedge dx_n)(p)$).

$\Leftarrow$) Let ω be a nowhere vanishing n-form on M. Choose any covering $\{U_\alpha, \varphi_\alpha\}$ by coordinate neighborhoods with local coordinates $x_1^\alpha, \cdots, x_n^\alpha$ s.t. $\omega|_{U_\alpha} = \lambda^\alpha(x)\, dx_1^\alpha \wedge \cdots \wedge dx_n^\alpha$. We can choose $x_1^\alpha, \cdots, x_n^\alpha$ s.t. $\lambda^\alpha > 0$. If $U_\alpha \cap U_\beta \neq \phi$, then $\omega|_{U_\alpha \cap U_\beta} = \lambda^\alpha dx_1^\alpha \wedge \cdots \wedge dx_n^\alpha = \lambda^\alpha \det\left(\frac{\partial x_j^\alpha}{\partial x_i^\beta}\right) dx_1^\beta \wedge \cdots \wedge dx_n^\beta = \lambda^\beta dx_1^\beta \wedge \cdots \wedge dx_n^\beta$.
$\therefore \lambda^\alpha \det\left(\frac{\partial x_j^\alpha}{\partial x_i^\beta}\right) = \lambda^\beta$ and hence $\det\left(\frac{\partial x_j^\alpha}{\partial x_i^\beta}\right) > 0$ and M is orientable.

* Integration in $\mathbb{R}^n$: the Riemann integral.

Change of variable formula : $\int_a^b f(x)dx = \int_c^d f(g(x))g'(x)dx$, $g(c)=a$, $g(d)=b$.

φ: diffeomorphism of $D \subset \mathbb{R}^n$ onto $\varphi(D)$. $J(\varphi)$: determinant of the Jacobian matrix of φ: $J\varphi = \det\left(\frac{\partial \varphi_j}{\partial r_i}\right)$. Then $\int_{\varphi(A)} f = \int_A f \circ \varphi\, |J\varphi|$, f: bounded continuous.

ω: n-form on $D \subset \mathbb{R}^n$. $\exists f \in C^\infty(D)$ s.t. $\omega = f dr_1 \wedge \cdots \wedge dr_n$. Define $\int_A \omega = \int_A f$, $A \subset D$. Change of variable formula $\Rightarrow$ $\int_{\varphi(A)} \omega = \pm \int_A \varphi^*(\omega)$, "+" if φ is orientation preserving, "−" if φ is orientation reversing.

$\because \varphi^*(\omega) = f\circ\varphi\, \varphi^*(dr_1) \wedge \cdots \wedge \varphi^*(dr_n) = f\circ\varphi\left(\sum \frac{\partial \varphi_1}{\partial r_i} dr_i\right) \wedge \cdots \wedge \left(\sum \frac{\partial \varphi_n}{\partial r_i} dr_i\right)$
$= f\circ\varphi \det\left(\frac{\partial \varphi_j}{\partial r_i}\right) dr_1 \wedge \cdots \wedge dr_n$

* Integration in M. ω: n-form on M, assume $\mathrm{supp}(\omega) \subset (U, \varphi)$.

$\varphi^{-1*}(\omega) = f dr_1 \wedge \cdots \wedge dr_n$. Define $\int_M \omega = \int_{\varphi(U)} f$: well-defined. This is because if $\mathrm{supp}(\omega) \subset (V, \psi)$, then $\varphi^{-1*}(\omega) = (\psi \circ \varphi^{-1})^* \psi^{-1*}(\omega)$ and

$\int_{\varphi(U)} \varphi^{-1*}(\omega) = \int_{\varphi(U)} (\psi\circ\varphi^{-1})^* \psi^{-1*}(\omega) = \int_{\psi\circ\varphi^{-1}(\varphi(U))} \psi^{-1*}(\omega) = \int_{\psi(U)} \psi^{-1*}(\omega)$

∵ If $\psi^{-1*}(\omega) = g\, dr_1 \wedge \cdots \wedge dr_n$, then $\int_{\varphi(U)} f = \int_{\psi(V)} g$. ∴ well-defined.

What if $supp(\omega) \not\subset$ coordinate neighborhood? Choose a covering $\{U_\alpha, \varphi_\alpha\}$ of $supp(\omega)$ and a partition of unity $\{f_i\}$ subordinate to $\{U_\alpha, \varphi_\alpha\}$. Since $\sum f_i \equiv 1$, $\omega = \sum_i f_i \omega$. Define $\int_M \omega = \sum_i \int_M f_i \omega$. well-defined?

Let $\{g_j\}$ be another partition of unity subordinate to a cover $\{V_\beta, \psi_\beta\}$.

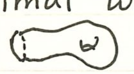
$\int_M \omega = \sum_i \int_M f_i \omega = \sum_{i,j} \int_M g_j f_i \omega = \sum_j \int_M g_j (\sum_i f_i \omega) = \sum_j \int_M g_j \omega$.

<u>Definition</u> A C^∞ <u>manifold with boundary</u> is a Hausdorff space M with a countable basis of open sets and a differentiable structure $\mathcal{F}$ in the following sense: $\mathcal{F} = \{U_\alpha, \varphi_\alpha\}$ consists of a family of open sets U_α of M each with a homeomorphism φ_α onto an open subset of $H^n = \{(x_1, \cdots, x_n) \in \mathbb{R}^n : x^n \geq 0\}$ such that (topologized as a subspace of $\mathbb{R}^n$)

(1) the U_α cover M;

(2) if $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) are elements of $\mathcal{F}$, then $\varphi_\beta \circ \varphi_\alpha^{-1}$ and $\varphi_\alpha \circ \varphi_\beta^{-1}$ are diffeomorphisms of $\varphi_\alpha(U_\alpha \cap U_\beta)$ and $\varphi_\beta(U_\alpha \cap U_\beta)$, open subsets of H^n;

(3) $\mathcal{F}$ is maximal with respect to the properties (1) and (2).

ex) B^n, S^2_+, 

<u>Theorem</u> If M^n is a C^∞ manifold with boundary, then the differentiable structure of M determines a C^∞-differentiable structure of dimension $n-1$ on the subspace ∂M of M. The inclusion $i : \partial M \to M$ is an embedding.

pf) $(U_\alpha, \varphi_\alpha) \in \mathcal{F} \Rightarrow \{(U_\alpha \cap \partial M, \varphi_\alpha|_{U_\alpha \cap \partial M})\}$: differentiable structure for ∂M

<u>Theorem</u> Let M be an oriented manifold with boundary. Then ∂M is orientable and the orientation of M determines an orientation of ∂M.

<u>Theorem</u> (Stokes' Theorem) Let M^n be an oriented compact manifold. and let ω be a smooth $(n-1)$-form of compact support on M. Then $\int_M d\omega = \int_{\partial M} \omega$.

pf) Suppose $supp(\omega) \subset (U, \varphi)$, $\varphi(U) \subset$ cube.

$\varphi^{-1*}(\omega) = \sum_j (-1)^{j-1} \lambda_j\, dr_1 \wedge \cdots \wedge \widehat{dr_j} \wedge \cdots \wedge dr_n$.

$\varphi^{-1*}(d\omega) = d\varphi^{-1*}(\omega) = \sum \frac{\partial \lambda_j}{\partial r_j} dr_1 \wedge \cdots \wedge dr_n$.

$\therefore \int_M d\omega = \sum_j \int_0^1 \cdots \int_0^1 \frac{\partial \lambda_j}{\partial r_j} dr_1 \cdots dr_n = \int_0^1 \cdots \int_0^1 [\lambda_j(r_1, \cdots, r_{j-1}, 1, r_{j+1}, \cdots, r_n)$
$- \lambda_j(r_1, \cdots, r_{j-1}, 0, r_{j+1}, \cdots, r_n)] dr_1 \cdots \widehat{dr_j} \cdots dr_n$

$= 0$ if $\mathrm{supp}(\omega) \cap \partial M = \phi$, and $\overset{\text{otherwise}}{=} -\int_0^1 \cdots \int_0^1 \lambda_n(r_1, \cdots, r_{n-1}, 0) dr_1 \cdots dr_{n-1}$.

$\varphi^{-1*}(\omega|_{\partial M}) = (-1)^{n-1} \lambda_n(r_1, \cdots, r_{n-1}, 0) dr_1 \wedge \cdots \wedge dr_{n-1}$.

$\therefore \int_{\partial M} \omega = (-1)^{n-1} \int_0^1 \cdots \int_0^1 \lambda_n(r_1, \cdots, r_{n-1}, 0) dr_1 \cdots dr_{n-1}$

$\therefore \int_M d\omega = (-1)^n \int_{\partial M} \omega = \int_{\partial M} \omega$ after change of orientation in such a way that $v_1, \cdots, v_{n-1}$ is an oriented basis of $T_p \partial M \overset{\text{def}}{\iff} \eta, v_1, \cdots, v_{n-1}$ is an oriented basis of $T_p M$, where η is an outer vector to ∂M at p.

* Integration on a Riemannian manifold

Metric $\langle\ ,\ \rangle$ gives a canonical isomorphism $\mathfrak{X}(M) \cong E^1(M)$. $X \in \mathfrak{X}(M) \Rightarrow \tilde{X} \in E^1(M)$, $\tilde{X}(Y) = \langle X, Y \rangle$. $\omega \in E^1(M) \Rightarrow \tilde{\omega} \in \mathfrak{X}(M)$, $\langle \tilde{\omega}, Y \rangle = \omega(Y)$.

$x_1, \cdots, x_n$: local coordinates on $M \Rightarrow \frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n} \in \mathfrak{X}(M) \xRightarrow{\text{Gram-Schmidt}}$ Orthonormal vector fields $X_1, \cdots, X_n \in \mathfrak{X}(M) \Rightarrow \tilde{X}_1, \cdots, \tilde{X}_n \in E^1(M)$. Suppose $\tilde{X}_1 \wedge \cdots \wedge \tilde{X}_n$ determines the orientation of M. $\tilde{X}_1 \wedge \cdots \wedge \tilde{X}_n$ agrees on overlaps of coordinate domains. $\therefore \tilde{X}_1 \wedge \cdots \wedge \tilde{X}_n$ is a globally defined nowhere-vanishing n-form ω on M. ω is called the <u>volume form</u> of the oriented Riemannian manifold M. Its integral over M is the <u>volume</u> of M.

$g_{ij} = \langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \rangle = \langle \sum \alpha_i^k X_k, \sum \alpha_j^l X_l \rangle = \sum_k \alpha_i^k \alpha_j^k$. $\left(\frac{\partial}{\partial x_i} = \sum \alpha_i^k X_k \right)$

$\therefore (g_{ij}) = A^t A$, $A = (\alpha_i^k)$. $\omega\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right) = \omega\left(\sum \alpha_1^k X_k, \cdots, \sum \alpha_n^k X_k\right)$
$= \det(\alpha_i^k) = \sqrt{\det(g_{ij})}\, dx_1 \wedge \cdots \wedge dx_n \left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right)$. $\therefore \omega = \sqrt{g}\, dx_1 \wedge \cdots \wedge dx_n$,
$g := \det(g_{ij})$.