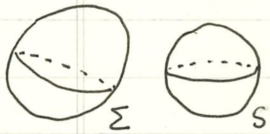
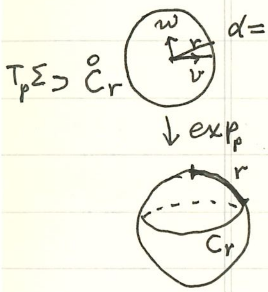
in such a way that π preserves the orientation. M: compact $\Rightarrow K \geq \delta > 0$ on $\tilde{M} \Rightarrow \tilde{M}$: compact. $k: \tilde{M} \to \tilde{M}$ a covering transformation, $\pi \circ k = \pi$.
$\Rightarrow k$ is an isometry and preserves the orientation. n: even, Thm $\Rightarrow k$ has a fixed point. $\Rightarrow k$: identity. $\Rightarrow \pi_1(M) = 0$ $\because$ the group of covering transformations $\approx \pi_1(M)$.

(b) $\bar{M}$: orientable double cover of M, with covering metric. Let k be a covering transformation of $\bar{M}$, $k \neq id$. k is an isometry which reverses the orientation of $\bar{M}$. n: odd, Thm $\Rightarrow k$ has a fixed point. $\Rightarrow k = id$. ※

ex) $P^2 \times P^2$ cannot have a metric with positive sectional curvature.
$S^1 \times P^2$ cannot either.

ex) 

$K_\Sigma \leq K_{S^2}$. Rauch $\Rightarrow |J| \geq \psi$, J: Jacobi field on Σ.
$\psi = |\bar{J}|$, $\bar{J}$: Jacobi field on S^2, $|\bar{J}|(0) = 0 = \psi(0)$, $|J|'(0) = \psi'(0)$. C_r: circle of radius r in Σ, $\bar{C}_r \subset S^2$. $\mathring{C}_r \subset T_p\Sigma$.



α = length expansion ratio of exp_p on $\mathring{C}_r$.
$J = d\exp(rw)$ $\therefore$ $\alpha = |J|/|rw| = |J|/r$.
$L(C_r) = \int_{\mathring{C}_r} \alpha\, ds = \int_{\mathring{C}_r} \frac{|J|}{r} ds = \int_{\mathring{C}_1} |J| d\theta \geq \int_{\mathring{C}_1} \psi\, d\theta = L(\bar{C}_r)$
D_r: disk of radius r in Σ, $\bar{D}_r \subset S^2$, $\mathring{D}_r \subset T_p\Sigma$.
$A(D_r) = \int_0^r L(C_r) dr \geq \int_0^r L(\bar{C}_r) dr = A(\bar{D}_r)$.

Differential Forms

u, v: vectors in $\mathbb{R}^3$. $u \cdot v$, $u \times v$ $\Rightarrow$ metric tensor, differential form.
$(u \times v) \cdot w = \det(u, v, w)$ = volume of parallelepiped : differential form.
determinant: multilinear, alternating. metric: bilinear, symmetric

<u>Definition</u> An alternating $C^\infty(M)$-multilinear map $\omega: \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{k \text{ copies}} \to C^\infty(M)$ of order k on C^∞ manifold M will be called an <u>exterior differential form of degree k</u> (or sometimes simply k-form).

1. ω is called <u>alternating</u> if

$$\omega(X_{\pi(1)}, \cdots, X_{\pi(k)}) = (\operatorname{sgn} \pi)\, \omega(X_1, \cdots, X_k) \quad (X_1, \cdots, X_k \in \mathfrak{X}(M)),$$

for all permutations π in the permutation group S_k on k letters. $\mathrm{Sgn}\,\pi$ is the sign of the permutation π (+1 if π is even, −1 if π is odd).

2. ω is $C^\infty(M)$-multilinear if
$$\omega(X_1,\cdots,X_{i-1},fX+gY,X_{i+1},\cdots,X_k)=f\omega(X_1,\cdots,X_{i-1},X,X_{i+1},\cdots,X_k)+g\omega(X_1,\cdots,X_{i-1},Y,X_{i+1},\cdots,X_k)$$
whenever $f,g\in C^\infty(M)$ and $X_1,\cdots,X_{i-1},X,Y,X_{i+1},\cdots,X_k\in\mathfrak{X}(M)$.
3. ω is called C^∞ if $\omega(X_1,\cdots,X_k)\in C^\infty(M)$ whenever $X_1,\cdots,X_k\in\mathfrak{X}(M)$.
4. $E^k(M)$ denotes the set of all smooth exterior differential form of degree k on M. Given $\omega\in E^k(M)$ and $\varphi\in E^l(M)$, the <u>exterior product</u> (or <u>wedge product</u>) of ω and φ, denoted $\omega\wedge\varphi$, is defined by $\omega\wedge\varphi(X_1,\cdots,X_{k+l})=\sum_{k,l\,\text{shuffles}}(\mathrm{sgn}\,\pi)\,\omega(X_{\pi(1)},\cdots,X_{\pi(k)})\,\varphi(X_{\pi(k+1)},\cdots,X_{\pi(k+l)})$.
Here a permutation $\pi\in S_{k+l}$ is called a "k,l shuffle" if $\pi(1)<\cdots<\pi(k)$ and $\pi(k+1)<\cdots<\pi(k+l)$. In fact, $\omega\wedge\varphi=(-1)^{kl}\varphi\wedge\omega\in E^{k+l}(M)$
If $\eta_1,\cdots,\eta_k\in E^1(M)$ and $\omega=\eta_1\wedge\eta_2\wedge\cdots\wedge\eta_k$, then
$$\omega(X_1,\cdots,X_k)=\det(\eta_i(X_j)).$$
5. $f\in C^\infty(M)\Rightarrow df\in E^1(M)$, $df(X)=Xf$, $X\in\mathfrak{X}(M)$.
$X\in\mathfrak{X}(M)\Rightarrow$ The dual 1-form ω of X is defined by $\omega(Y)=\langle X,Y\rangle$.
$x_1,\cdots,x_n$: local coordinates on $U\subset M^n\Rightarrow$ For any 1-form $\omega\in E^1(M)$
$\omega=\sum a_i\,dx_i$ on U. Remember $X\in\mathfrak{X}(M)\Rightarrow X=\sum b_i\frac{\partial}{\partial x_i}$ on U.
$f\in C^\infty(M)\Rightarrow df=\sum\frac{\partial f}{\partial x_i}dx_i$ on U.
6. $f\in C^\infty(M)\Rightarrow df\in E^1(M)$. The 1-form df is called the <u>exterior derivative</u> of the 0-form $f\in E^0(M)$. This <u>exterior differentiation operator</u> d has an extension to $E^k(M)$, $k>0$.

<u>Theorem</u> There exists a unique $\mathbb{R}$-linear map $d: E^k(M)\to E^{k+1}(M)$ s.t.
(1) If $f\in E^0(M)=C^\infty(M)$, then df is the differential of f.
(2) If $\omega\in E^r(M)$ and $\varphi\in E^s(M)$, then $d(\omega\wedge\varphi)=d\omega\wedge\varphi+(-1)^r\omega\wedge d\varphi$.
(3) $d^2=0$.

pf) (A) If d exists and $g,f_1,\cdots,f_r\in C^\infty(M)$, then (1)–(3) imply that for $\omega=g\,df_1\wedge\cdots\wedge df_r$

we must have $d\omega = dg \wedge df_1 \wedge \cdots \wedge df_r$. Suppose M is covered by coordinate functions $x_1, \cdots, x_n$. Then d must be given by

(*) $d(\sum a_{i_1 \cdots i_r} dx_{i_1} \wedge \cdots \wedge dx_{i_r}) = \sum da_{i_1 \cdots i_r} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_r}$, where $da_{i_1 \cdots i_r} = \sum_{j=1}^{n} \frac{\partial a_{i_1 \cdots i_r}}{\partial x_j} dx_j$

and $1 \le i_1 < i_2 < \cdots < i_r \le n$. Therefore, if defined at all, d is unique in this case.

The d defined by (*) is $\mathbb{R}$-linear and trivially satisfies (1). For $f \in C^\infty(M)$, $d(df) = \sum d(\frac{\partial f}{\partial x_i}) \wedge dx_i = \sum_{ij} \frac{\partial^2 f}{\partial x_j \partial x_i} dx_j \wedge dx_i = 0$. $\therefore$ (3) holds.

$\omega = a\, dx_{i_1} \wedge \cdots \wedge dx_{i_r}$ and $\varphi = b\, dx_{j_1} \wedge \cdots \wedge dx_{j_s}$. Then

$d[(a\, dx_{i_1} \wedge \cdots \wedge dx_{i_r}) \wedge (b\, dx_{j_1} \wedge \cdots \wedge dx_{j_s})] = d(ab)(dx_{i_1} \wedge \cdots \wedge dx_{i_r}) \wedge (dx_{j_1} \wedge \cdots \wedge dx_{j_s})$

$= [(da)b + a(db)] \wedge (dx_{i_1} \wedge \cdots \wedge dx_{i_r}) \wedge (dx_{j_1} \wedge \cdots \wedge dx_{j_s})$

$= (da \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_r}) \wedge (b\, dx_{j_1} \wedge \cdots \wedge dx_{j_s}) + (-1)^r (a\, dx_{i_1} \wedge \cdots \wedge dx_{i_r}) \wedge (db \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_s}) \Rightarrow (3)$.

(B) Suppose that d with properties (1)-(3) is defined and that $U \subset M$ is a coordinate neighborhood with coordinates $x_1, \cdots, x_n$. (*) uniquely defines d_U. Show that the restriction of $d\omega$ to U is equal to d_U applied to ω restricted to U: $(d\omega)_U = d_U \omega_U$.

(C) If d satisfying (1)-(3) exists, it is unique. Cover M by coordinate neighborhoods $\{U_\alpha, \varphi_\alpha\}$ and define $(d\omega)_{U_\alpha} = d_{U_\alpha} \omega_{U_\alpha}$. Let $U = U_\alpha \cap U_\beta$ and show $(d_{U_\alpha} \omega_{U_\alpha})_U = d_U \omega_U = (d_{U_\beta} \omega_{U_\beta})_U$.

If $\psi: M \to N$ is a smooth map, $d\psi$ is the differential of ψ at $p \in M$ which is a linear map $d\psi: T_pM \to T_{\psi(p)}N$ defined by $d\psi(v)(g) = v(g \circ \psi)$, $v \in T_pM$, $g \in C^\infty(N)$. If ω is a form on N, then we can pull ω back to a form on M by setting $\psi^*(\omega)|_p = \psi^*(\omega|_{\psi(p)})$. More precisely, $\psi^*(\omega)(X_1, \cdots, X_k)(p) = \omega_{\psi(p)}(d\psi(X_1(p)), \cdots, d\psi(X_k(p)))$ for $\omega \in E^k(N)$ and for vector fields $X_1, \cdots, X_k$ on M.

Theorem ψ^* commutes with d; that is, $d(\psi^*(\omega)) = \psi^*(d\omega)$, $\omega \in E^k(M)$.

pf) $f \in C^\infty(M) = E^0(M) \Rightarrow \psi^*(f) = f \circ \psi \in C^\infty(M)$. $\psi^*(df_{\psi(p)})(v) = df(d\psi(v)) = d\psi(v)(f)$ $= v(f \circ \psi) = d(f \circ \psi)_p(v)$, $v \in T_pM$. $\therefore \psi^*(df) = d(f \circ \psi) = d(\psi^*(f))$.

Let $\omega \in E^k(M)$, $p \in M$. Choose a coordinate system $(U, x_1, \cdots, x_n)$ about $\psi(p)$ and a neighborhood V of p s.t. $\omega|U = \sum a_{i_1 \cdots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$. It follows that $\psi^*(\omega)|_V = \sum a_{i_1 \cdots i_k} \circ \psi \, d(x_{i_1} \circ \psi) \wedge \cdots \wedge d(x_{i_k} \circ \psi)$. Hence $\psi^*(\omega) \in E^k(M)$.