ex) T^3 and $S^2 \times S^1$ can have no positive Ricci curvature.

Rauch Comparison Theorem

Let M^n, M_0^{n+k} be Riemannian manifolds and let $\gamma, \gamma_0 : [0,\ell] \to M, M_0$ be normal geodesics, and set $\gamma' = T$, $\gamma_0' = T_0$. Assume $K_{M_0} \geqq K_M$. Assume further that for no $t \in [0,\ell]$ is $\gamma_0(t)$ conjugate to $\gamma_0(0)$ along γ_0. Let V, V_0 be Jacobi fields along γ, γ_0 s.t. $V(0), V_0(0)$ are perpendicular to γ, γ_0 and

$\|V(0)\| = \|V_0(0)\|$, $\langle T, V'(0)\rangle = \langle T_0, V_0'(0)\rangle$, $\|V'(0)\| = \|V_0'(0)\|$.

Then for all $t \in [0,\ell]$, $\|V(t)\| \geqq \|V_0(t)\|$.

Rauch Comparison Theorem (Simpler Version)

Assume $K_M \leqq a$.

(1) If J is a Jacobi field along a unit-speed geodesic $\gamma|_{[0,\ell]}$ and J is perpendicular to γ, then $\|J\|'' + a\|J\| \geqq 0$ along γ.

(2) If ψ is a solution on $[0,\ell]$ of $\psi'' + a\psi = 0$, $\psi(0) = \|J\|(0)$, $\psi'(0) = \|J\|'(0)$, and $\psi \neq 0$ on $(0,\ell)$, then $\left(\frac{\|J\|}{\psi}\right)' \geqq 0$ and $\|J\| \geqq \psi$ on $(0,\ell)$.

($\psi = \|J_a\|$, J_a is the Jacobi field along $\bar{\gamma}$ in $\bar{M}$ with $K_{\bar{M}} \equiv a$.)

pf) $\|J\|' = \frac{\langle J, \nabla_T J\rangle}{\|J\|}$, $\|J\|'' = \|J\|^{-1}\{\|\nabla_T J\|^2 + \langle R(T,J)T, J\rangle\} - \|J\|^{-3}\langle J, \nabla_T J\rangle^2$

$\geqq -a\|J\| + \|J\|^{-3}\{\|\nabla_T J\|^2\|J\|^2 - \langle J, \nabla_T J\rangle^2\} \geqq -a\|J\|$.

(2) $\left(\frac{\|J\|}{\psi}\right)' = \frac{\|J\|'\psi - \|J\|\psi'}{\psi^2}$. $F := \|J\|'\psi - \|J\|\psi'$, $F(0) = 0$.

Then $F' = \|J\|''\psi - \|J\|\psi'' \geqq -a\|J\|\psi + \|J\|a\psi = 0$.

Theorem (Cartan–Hadamard) Let M be complete and $K_M \leqq 0$. Then for any $p \in M$, $\exp_p : T_pM \to M$ is a covering map. Hence the universal covering space of M is diffeomorphic to $\mathbb{R}^n$, and the homotopy groups $\pi_i(M)$ vanish for $i > 1$.

Lemma If $\varphi : M^n \to N^n$ is a local isometry and M is complete, then φ is a covering map.

pf of thm) Rauch $\Rightarrow$ M has no conjugate points. $\Rightarrow$ $d\exp_p$ is nonsingular. $\Rightarrow$ $\exp_p$ can pull back the metric of M on T_pM which makes it a local isometry. The lines through the origin of T_pM are geodesics because they are mapped by $\exp_p$ into geodesics on M. ∴ By Hopf-Rinow theorem T_pM is complete. ∴ Lemma ⇒ $\exp_p$ is a covering map. ∴ $\pi_i(\tilde{M}) = \pi_i(M)$ for $i \geq 2$ if $\tilde{M}$ covers M. Hence $\pi_i(M) = \pi_i(\mathbb{R}^n) = 0$.

<u>Corollary</u> Let M^n be complete, simply connected and have nonpositive curvature. Then M is diffeomorphic to $\mathbb{R}^n$.

ex) $S^2 \times S^2$ cannot have a metric with nonpositive curvature.

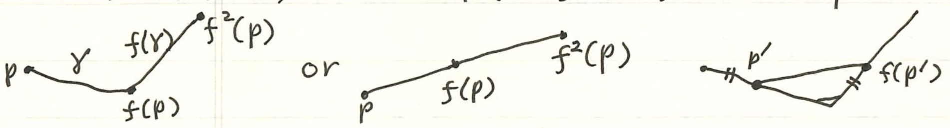
ex) Bonnet-Myers: $Ric(x, x) \geq (n-1)H \Rightarrow d(M) \leq \frac{\pi}{\sqrt{H}}$: sharp ∵ $M = S^n$.

$z = x^2 + y^2$: $H = 0$, noncompact.

Let M^n be complete with $Ric(x,x) \geq \frac{1}{r^2}$, $\forall$ unit x. If $d(M) = \pi r$, then M^n is isometric to $S^n(r)$. (S.Y. Cheng)

<u>Theorem</u> (Weinstein) Let f be an isometry of a compact oriented Riemannian manifold M^n. Suppose that M has positive sectional curvature and that f preserves the orientation of M if n is even, and reverses it if n is odd. Then f has a fixed point, i.e., there exists $p \in M$ with $f(p) = p$.

pf) Suppose that $f(q) \neq q$ for all $q \in M$. Let $p \in M$ s.t. $d(p, f(p))$ attains a minimum. There exists a normal minimizing geodesic $\gamma : [0, \ell] \to M$ with $\gamma(0) = p$, $\gamma(\ell) = f(p)$. Is $\gamma \cup f(\gamma)$ sharp or smooth at $\gamma(\ell)$?

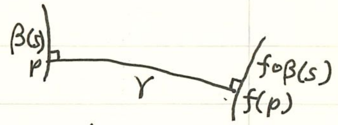


$d(p', f(p')) \leq d(p', f(p)) + d(f(p), f(p')) = d(p', f(p)) + d(p, p') = d(p, f(p))$.

Because $d(p, f(p))$ is minimum, $d(p', f(p')) = d(p', f(p)) + d(f(p), f(p'))$.

∴ $\gamma \cup f(\gamma)$ is a geodesic.

Let $\beta(s)$ be a geodesic with $\beta(0) = p$ s.t. $\beta'(0) \perp \gamma'(0)$. Then $f(\beta)$ is a geodesic from $f(p)$, that is, $f \circ \beta(0) = f(p)$ s.t. $(f \circ \beta)'(0) \perp \gamma'(\ell)$ because $(f \circ \gamma)'(0) = \gamma'(\ell)$. Is there a parallel vector field $e_1(t)$ along $\gamma(t)$ s.t. $e_1(0) = \beta'(0)$ and $e_1(\ell) = (f \circ \beta)'(0)$?



Let $\tilde{A} = P \circ df_p : T_pM \to T_pM$, where P is the parallel translation from along γ

$f(p) = \gamma(\ell)$ to p. Then $\tilde{A}$ is an isometry. Such parallel field $e_1(t)$ exists if $\tilde{A}(e_1(0)) = e_1(0)$.

Lemma Let A be an orthogonal linear transformation of $\mathbb{R}^{n-1}$ and suppose that $\det A = (-1)^n$. Then A leaves invariant some nonzero vector of $\mathbb{R}^{n-1}$.

pf) n: even $\Rightarrow \det(A - \lambda I)$: polynomial of odd degree. $\therefore A$ has a real eigenvalue, ± 1. Conjugate eigenvalues $a \pm ib \Rightarrow$ Their product is positive. $\det(A) = 1 \Rightarrow$ One eigenvalue equals 1.

n: odd $\Rightarrow \det(A) = -1 \Rightarrow \exists$ at least two real eigenvalues, one of which is positive, $= 1$.

$(f \circ \gamma)'(0) = \gamma'(\ell) \Rightarrow \tilde{A}(\gamma'(0)) = \gamma'(0)$. Let A be the restriction of $\tilde{A}$ to the orthogonal complement of $\gamma'(0)$, $\gamma'(0)^{\perp}$. Then A is orthogonal on $\mathbb{R}^{n-1}$. $\det A = \det \tilde{A} = \det(P \circ df_p) = (-1)^n$, $\because P$ preserves orientation. Lemma $\Rightarrow A$ leaves a vector v invariant. Let $e_1(t)$ be a unit parallel field along γ s.t. $e_1(\ ) = v$. Then $e_1(t) \perp \gamma(t)$.
$P \circ df_p(e_1(0)) = e_1(0) \Rightarrow df_p(e_1(0)) = e_1(\ell)$. β: geodesic, $\beta(0) = p$, $\beta'(0) = e_1(0) \Rightarrow f \circ \beta(s)$: geodesic, $f \circ \beta(0) = f(p) = \gamma(\ell)$, $(f \circ \beta)'(0) = e_1(\ell)$.

Let h be a variation of γ given by $h(s,t) = \exp_{\gamma(t)}(s e_1(t))$, $s \in (-\varepsilon, \varepsilon)$, $t \in [0, \ell]$. $h(s,0) = \beta(s) \Rightarrow h(s,\ell) = \exp_{f(p)}(s e_1(\ell)) = (f \circ \beta)(s)$. $V(t) = \frac{d}{ds} \exp_{\gamma(t)}(s e_1(t))\big|_{s=0} = e_1(t)$. $\therefore \nabla_{\gamma'} V = 0$. Therefore

$$\frac{\partial^2 L}{\partial s^2}(0) = \langle \nabla_V V, \gamma' \rangle \Big|_0^\ell + \int_0^\ell \langle \nabla_{\gamma'} V, \nabla_{\gamma'} V \rangle - \langle R(V,T)T, V \rangle$$

$V(s,0) = \frac{d}{ds} \exp_p(s e_1(0)) = \beta'(s)$, $V(s,\ell) = \frac{d}{ds} \exp_{f(p)}(s e_1(\ell)) = (f \circ \beta)'(s)$

$\therefore \nabla_V V(0,0) = 0$, $\nabla_V V(0,\ell) = 0$. $\therefore L''(0) < 0$: contradiction.

ex) $f : S^n \to S^n$: antipodal map.

Corollary (Synge) Let M^n be a compact manifold with positive sectional curvature. (a) If M^n is orientable and n is even, then M is simply connected. (b) If n is odd, then M^n is orientable.

pf) $\pi : \tilde{M} \to M$ universal cover. $\tilde{M}$ has the covering metric. Orient $\tilde{M}$