: The second variation formula. Valid for piecewise smooth variations.
In case the variation is through geodesics, $T\langle V,T\rangle$ and $T\langle W,T\rangle$ are constant.
Then if $\langle V,T\rangle$ or $\langle W,T\rangle$ vanishes at both endpoints, the last term drops out.
If V or W vanishes at the endpoints, or more generally, $\nabla_V W=0$, we get

$$\left.\frac{\partial^2 L}{\partial w\,\partial v}\right|_{(0,0)} = \int_a^b \langle \nabla_T V, \nabla_T W\rangle + \langle R(W,T)V,T\rangle.$$

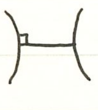
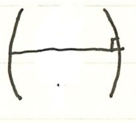
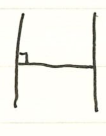
In this case the second variation depends only on the restrictions of V, W to γ.
We call the above integral the <u>index form</u> $I(V,W)$. : symmetric bilinear form on the space of piecewise smooth vector fields V, W along γ s.t. $\langle V,T\rangle\equiv\langle W,T\rangle\equiv 0$.
I is independent of the orientation of γ.
If I is positive definite on vector fields vanishing at $\gamma(a)$, $\gamma(b)$, then γ is a minimum among all nearby curves with the same endpoints.

<u>Proposition</u> Let I be defined on all piecewise smooth vector fields along γ which vanish at the endpoints. Then the null space of I is exactly the set of Jacobi fields along γ which vanish at $\gamma(a)$ and $\gamma(b)$. Specifically, V is a Jacobi field if and only if $I(V,W)=0$ for all W.

pf) Let f be a function vanishing at the endpoints and positive elsewhere.
$W := f(t)(-\nabla_T\nabla_T V + R(T,V)T) \Rightarrow V$ is a Jacobi field.

<u>Corollary</u> I has a nontrivial null space if and only if $\gamma(a)$ is conjugate to $\gamma(b)$ along γ. The dimension of the null space is the order of the conjugate point $\gamma(b)$.

pf) $J'(0) \longleftrightarrow \frac{d}{ds}\exp_{\gamma(0)}(T+sJ'(0))t$: linear isomorphism between the null space of $d\exp$ and the space of Jacobi fields along γ which vanish at the endpoints.

ex)  $L'=0$, $L''>0$  $L'=0$, $L''<0$  $L'=0$, $L''=0$

<u>Index Lemma</u> Let γ be a geodesic in M from p to q s.t. there are no points conjugate to p on γ. Let W be a piecewise smooth vector field on γ and V the unique Jacobi field s.t. $V(p)=W(p)=0$ and $V(q)=W(q)$. Then $I(V,V)\le I(W,W)$ and equality holds only if $V=W$.

pf) $\{V_i\}$: basis of T_qM. Extend each V_i to a Jacobi field along γ s.t. $V_i(p)=0$.
: unique $\because$ no conjugate points on γ. V_i are linearly independent except at p.
Since $V_i(p)=0$, $V_i = tA_i$ for t of $\gamma(t):[0,1]\to M$ and A_i : some vector field on γ.
$V_i'(p)=A_i\neq 0$. $\therefore \{A_i\}$ is linearly independent. $\therefore \exists\, g_i(t)$ s.t. $W=\sum g_i(t)A_i$.
Since $W_p=0$, $\exists\, C^\infty f_i$ s.t. $W=\sum f_iV_i$. Then $V=\sum f_i(1)V_i$

1) $I(V,V)=\langle V',V\rangle\big|_0^1+\int_0^1 -\langle V,\nabla_T\nabla_TV\rangle+\langle R(T,V)T,V\rangle=\langle V'(1),V(1)\rangle=\sum f_i(1)f_j(1)\langle V_i'(1),V_j(1)\rangle$

2) If V_i,V_j are Jacobi fields, then $\langle V_i',V_j\rangle-\langle V_i,V_j'\rangle=$ const. ($=0 \because V_i(p)=0$.)
$\because (\langle V_i',V_j\rangle-\langle V_i,V_j'\rangle)'=\langle V_i'',V_j\rangle+\langle V_i',V_j'\rangle-\langle V_i',V_j'\rangle-\langle V_i,V_j''\rangle$
$=\langle V_i'',V_j\rangle-\langle V_i,V_j''\rangle=\langle R(T,V_i)T,V_j\rangle-\langle V_i,R(T,V_j)T\rangle=0$.

Now $\nabla_TW=\sum f_i'V_i+f_iV_i' := A+B$.
$I(W,W)=\int\langle A,A\rangle+\langle A,B\rangle+\langle B,A\rangle+\langle B,B\rangle+\langle R(T,W)T,W\rangle$,
$\int\langle B,B\rangle=\sum\int f_if_j\langle V_i',V_j'\rangle=\sum\int f_if_j(\langle V_i',V_j\rangle'-\langle V_i'',V_j\rangle)$.
$=\sum f_i(1)f_j(1)\langle V_i'(1),V_j(1)\rangle-\int(f_i'f_j\langle V_i',V_j\rangle+f_if_j'\langle V_i',V_j\rangle+\langle R(T,W)T,W\rangle$.
By 1) the first term is $I(V,V)$. By 2) the second term is $\int\langle A,B\rangle$, third term $=\int\langle B,A\rangle$.
Hence $I(W,W)=I(V,V)+\int\langle A,A\rangle$. $\int\langle A,A\rangle\geq 0$ and "=" only if $W=V$.

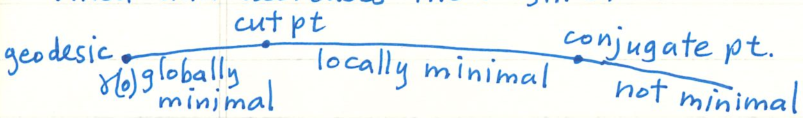
* $\gamma(t_0)$ is the first conjugate point of $\gamma(0)$ along γ. For W vanishing at $\gamma(0)$ and $\gamma(t):q$ with $t<t_0$, $L''=I(W,W)\geq I(V,V)=0$. $\therefore$ Geodesics minimize locally up to the first conjugate point among curves with the same end points.

<u>Corollary</u> Let $\gamma:[0,\infty)\to M$ be a geodesic, and let $\gamma(t_0)$ be conjugate to $\gamma(0)$.
Then $\gamma|[0,t]$ is not minimal for $t>t_0$.

pf) $\gamma(t_0)$: first point conjugate to $\gamma(0)$. J: nonzero Jacobi field along $\gamma([0,t_0])$ s.t.
$J(0)=J(t_0)=0$. Extend J to T on all of γ by $T(t)=0$ for $t\geq t_0$.

δ: small s.t. $\nexists$ conjugate points on $\gamma|[t_0-\delta,t_0+\delta]$.
Define V by $V=J$ on $[0,t_0-\delta]$, $V=$ Jacobi field W on $[t_0-\delta,t_0+\delta]$ s.t. $W(t_0-\delta)=J(t_0-\delta)$, $W(t_0+\delta)=0$,
$V=0$ on $[t_0+\delta,t]$. T is not a Jacobi field since it is not C^∞ on $[t_0-\delta,t_0+\delta]$.
Hence $I(V,V)<I(T,T)$ on $[t_0-\delta,t_0+\delta]$. Since $T=V$ outside this interval,

$I(V,V) < I(T,T) = 0$. $\therefore$ V induces a variation which keeps the endpoints fixed and decreases the length of γ.



* We define the <u>diameter</u> $d(M)$ of M to be the supremum of $d(p,q)$ for $p,q \in M$.

<u>Theorem</u> (Myers and Bonnet)

Let M^n be a complete Riemannian manifold. If

(1) (Myers) for all unit vectors x, $Ric(x,x) \geq (n-1)H$, or

(2) (Bonnet) $K_M \geq H$,

then every geodesic of length $\geq \frac{\pi}{\sqrt{H}}$ has conjugate points. Hence the diameter of M satisfies $d(M) \leq \frac{\pi}{\sqrt{H}}$.

pf) Fix a normal geodesic $\gamma : [0,\ell] \to M$. $\{E_i\}$: orthonormal basis of parallel fields along γ s.t. $E_n = \gamma' = T$. $W_i = \sin\frac{\pi t}{\ell} E_i(t)$ along γ. Then

$$I(W_i, W_i) = -\int_0^\ell \langle W_i, \nabla_T\nabla_T W_i + R(W_i,T)T\rangle dt$$
$$= \int_0^\ell \left(\sin^2\frac{\pi t}{\ell}\left(\frac{\pi^2}{\ell^2} - \langle R(E_i,T)T, E_i\rangle\right)\right) dt.$$

Thus if, for any i, $\langle R(E_i,T)T,E_i\rangle \geq H$ and $\ell \geq \frac{\pi}{\sqrt{H}}$, then $I(W_i,W_i) \leq 0$. (Bonnet)

Or If $R(T,T) \geq (n-1)H$ and $\ell \geq \frac{\pi}{\sqrt{H}}$, then one $I(W_i,W_i) \leq 0$. (Meyers)

$$\sum_{i=1}^{n-1} I(W_i,W_i) = \int_0^\ell \sin^2\frac{\pi t}{\ell}\left((n-1)\frac{\pi^2}{\ell^2} - Ric(T,T)\right)dt.$$

But if γ had no conjugate points, Index Lemma would imply that there is a Jacobi field J s.t. $I(J,J) < 0$ and J would vanish at $\gamma(0)$ and $\gamma(\ell)$: contradiction.

$\therefore$ γ has conjugate points, and γ is not minimal. (More directly, W_i gives a length decreasing variation.)

<u>Corollary</u> M: complete. If $Ric(x,x) \geq (n-1)H > 0$ for all unit vectors x and some $H > 0$, then M is compact and has finite fundamental group.

pf) $\tilde{M}$: universal cover of M. Since $\pi : \tilde{M} \to M$ is a local diffeomorphism, it induces a Riemannian structure on $\tilde{M}$, and the curvature tensor $\tilde{R}$ at $\tilde{p} \in \tilde{M}$ is isomorphic to R at $\pi(\tilde{p}) \in M$. $\therefore$ Myers' theorem $\Rightarrow d(\tilde{M}) \leq \frac{\pi}{\sqrt{H}}$. So $\tilde{M}$ is compact. $\therefore$ $\pi^{-1}(p)$ has finite cardinality and $\underset{1\text{-}1}{\cong}$ $\pi_1(M)/\pi_*\pi_1(\tilde{M})$: coset space.